FUNCTIONS
OF A
COMPLEX VARIABLE

BY

R. N. JAIN

B.A. (Hons.), M.A.

Jr. Professor of Mathematics,

HOLKAR COLLEGE, INDORE.

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PREFACE

This book is based on the lectures which the author has given during the last ten years. Subject to the limitations of the course, an attempt has been made to combine simple and clear exposition with rigour. The book, though written for the M. A. and M. Sc. students of Agra, Rajasthan and Vikram Universities, will be found useful by the students of other universities also.

I am indebted to many writers on the subject and their names are given in the bibliography at the end. This will also enable the reader to refer to these sources in case he desires to extend his knowledge.

Suggestions for improvement and notices of error will be gratefully acknowledged.

Holkar College, Indore.

AUTHOR

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BIBLIOGRAPHY

1. Caratheodory, C. — Canformal Mapping
 Theory of Functions
2. Churchill, R.V. — Complex Variables
3. Copson E.T. — Theory of Functions of a Complex Variable
4. Dienes, P. — Taylor Series
5. Goursat, E. — Mathematical Analysis Vol. II, Part I
6. Green, S.L. — Theory and Use of the Complex Variable
7. Hardy, G.H. — Pure Mathematics
8. Knopp, K. — Theory of Functions, Parts I and II
 Problem Books, Parts I and II
9. MacRobert, T.M. — Functions of a Complex Variable
10. Nehari, Z. — Conformal Mapping
11. Phillips, E.G. — Functions of a Complex Variable
12. Titchmarsh, E.C. — Theory of Functions
13. Townsend, E.J. — Functions of a Complex Variable
14. Watson, G.N. — Complex Integration and Cauchy's Theorem
15. Whittaker, E.T. and Watson, G.N. — Modern Analysis

CHAPTER I

COMPLEX NUMBERS

1.1. Definitions. The complex number $z=x+iy$ is defined by the *ordered pair* (x, y) of real numbers x and y, that satisfies certain laws of operation. [The number-pair (a, b) is said to be ordered if the number-pairs (a, b) and (b, a) are distinct, unless $a=b$.]

The number i is called the *imaginary unit* and satisfies the fundamental laws of algebra, associative, commutative and distributive, except that its square is -1.

As this notation suggests, when $y=0$, the complex number z becomes the real number x, *i. e.*, complex numbers include real numbers : $x+i.\ o=x$.

As a consequence, the laws of operation that are to be prescribed must reduce to laws of operation on real numbers when $y=0$.

When $x=0$, the number z is said to be purely imaginary. The real numbers x and y are known as the *real and imaginary components* (or *parts*) of z. In symbols $\mathbf{Re}\,(z)=x$, $\mathbf{Im}\,(z)=y$, where Re (z) stands for the real parts of z and Im (z) for the imaginary part of z.

Note 1. In the expression $x+iy$, the sign $+$ does not indicate addition as understood so far. The symbol i denotes a complex number represented by the ordered pair of real numbers $(0, 1)$. Also two real number are always required to express a single complex number. The notation $x+iy$, though suggesting addition, is used as a matter of convenience.

Note 2. The name *imaginary numbers* does not imply that there is anything unreal about them, or that these numbers live in a precarious condition in some people's imagination. This unfortunate name has, however, led many students of elementary mathematics to suspect the whole subject of complex variables.

1.2. Definition of Equality. Two complex numbers $z_1=(x_1, y_1)$ and $z_2=(x_2, y_2)$ are equal if, and only if, $x_1=x_2$, $y_1=y_2$.

In particular, since $0=(0, 0)$, a complex number z is zero if, and only if, its real and imaginary parts both vanish :

$$(x, y)=0 \text{ implies } x=0=y.$$

Note. The phrases 'greater than' or 'less than' have no meaning when applied to complex numbers. The only relation of magnitude that can exist between two complex numbers is that of equality.

1·21. The **modulus** of z, written $|z|$, is defined to be $\sqrt{x^2+y^2}$. It follows from the definition that $|z|=0$, if and only if, $x=0,\ y=0$.

1.3. **Fundamental Operations.*** If $z_1=(x_1, y_1),\ z_2=(x_2, y_2)$; we have the following definitions :

I. **Addition.** $z_1+z_2=(x_1+x_2\ y_1+y_2)$.

In particular, if $z_1=(x_1, 0)$ and $z_2=(0, y_1)$,

$$(x_1, 0)+(0, y_1)=(x_1, y_1).$$

II. **Subtraction.** $z_1-z_2=(x_1-x_2, y_1-y_2)$.

We define $-z$ as $(-x, -y)$, where $z=(x, y)$

In particular, when $x=0$, we have $-iy=(0, -y)$, since $z=x+iy$.

III. **Multiplication.** $z_1z_2=(x_1x_2-y_1y_2, x_1y_2+x_2y_1)$.

In particular, if $x_1=x_2=0$ and $y_1=y_2=1$, we have

$$i^2=(-1, 0)=-1.$$

If the fundamental operations are defined as above, we easily see that the complex numbers satisfy the fundamental laws of algebra of real numbers ; hence their algebra will be identical in form though not in meaning, with the algebra of real numbers. For example, the commutative law of multiplication holds :

$$z_1z_2=(x_1x_2-y_1y_2,\ x_1y_2+x_2y_1)$$
$$=(x_2x_1-y_2y_1,\ x_2y_1+x_1y_2)=z_2z_1.$$

The other laws can be similarly proved.

Note. Originally the algebra of complex numbers was introduced by regarding the number $x+iy$ as the sum of the number x and iy and by making use of $i^2=-1$ in the usual algebraic operations. The definition of a complex number as an ordered pair of real numbers clearly shows that this algebra need not depend upon the use of mysterious 'imaginary' numbers. The important thing to realize

* The knowledge of the fundamental operations with real numbers is assumed.

is that a complex number is in reality a pair of two real number and the customary algebraic operations can be defined for these number-pairs.

1.31. Zero Products. *If a product of two complex numbers be zero, then one of them is zero.*

For, let $(x_1,y_1)\times(x_2,y_2)=0$; then $(x_1x_2-y_1y_2,\ x_1y_2+x_2y_1)=0$. Hence, by the definition of equality

$$x_1x_2-y_1y_2=0, \quad \text{and} \quad x_1y_2+x_2y_1=0 ;$$

$$\therefore \quad x_1\,(x_2^2+y_2^2)=0, \quad \text{and} \quad y_1(x_2^2+y_2^2)=0.$$

If $x_2^2+y_2^2\neq 0$, it follows that $x_1=0=y_1$, and, $\therefore\ (x_1,y_1)=0$.

If $x_2^2+y_2^2=0$, then, since $x_2,\ y_2$ are real, $x_2=0, y_2=0$, and hence $(x_2,\ y_2)=0$.

1.32. Definition of division. If $z_1=(x_1,y_1)$, $z_2=(x_2,y_2)$, $z_3=(x_3,y_3)$; then, from the equation $z_2z_3=z_1$, we have

$$(x_2x_3-y_2y_3,\ x_2y_3+x_3y_2)=(x_1,y_1),$$

so that $x_2x_3-y_2y_3=x_1$ and $x_2y_3+x_3y_2=y_1$.

On solving for x_3 and y_3, $\quad x_3=\dfrac{x_1x_2+y_1y_2}{x_2^2+y_2^2}, \quad y_3=\dfrac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$.

provided that $|\ z_2\ |\neq 0$. Hence, if $|\ z_2\ |\neq 0$, there is a unique solution, and $z_3=(x_3,\ y_3)$ is the quotient of $\dfrac{z_1}{z_2}$.

In particular, if we write, as before, 1 for (1,0), then from above, if $z_1=(1,0)$, we have $\dfrac{1}{z_2}=\left(\dfrac{x_2}{x_2^2+y_2^2},\dfrac{-y_2}{x_2^2+y_2^2}\right)$.

This defines the *reciprocal* of z_2.

1.4. Geometrical Representation. The definition of a complex number as an ordered pair of real numbers immediately suggests a geometrical representation of complex numbers which is due to Gauss and Argand.

In the rectangular coordinate system $X'OX$, $Y'OY$, the point $P\,(x,y)$ is uniquely determined by the real number-pair x,y. Hence, the point P can represent the complex number $z=(x,y)$. The point P is called the point z and the figure is then called the **Argand Diagram** (or **Argand Plane**).

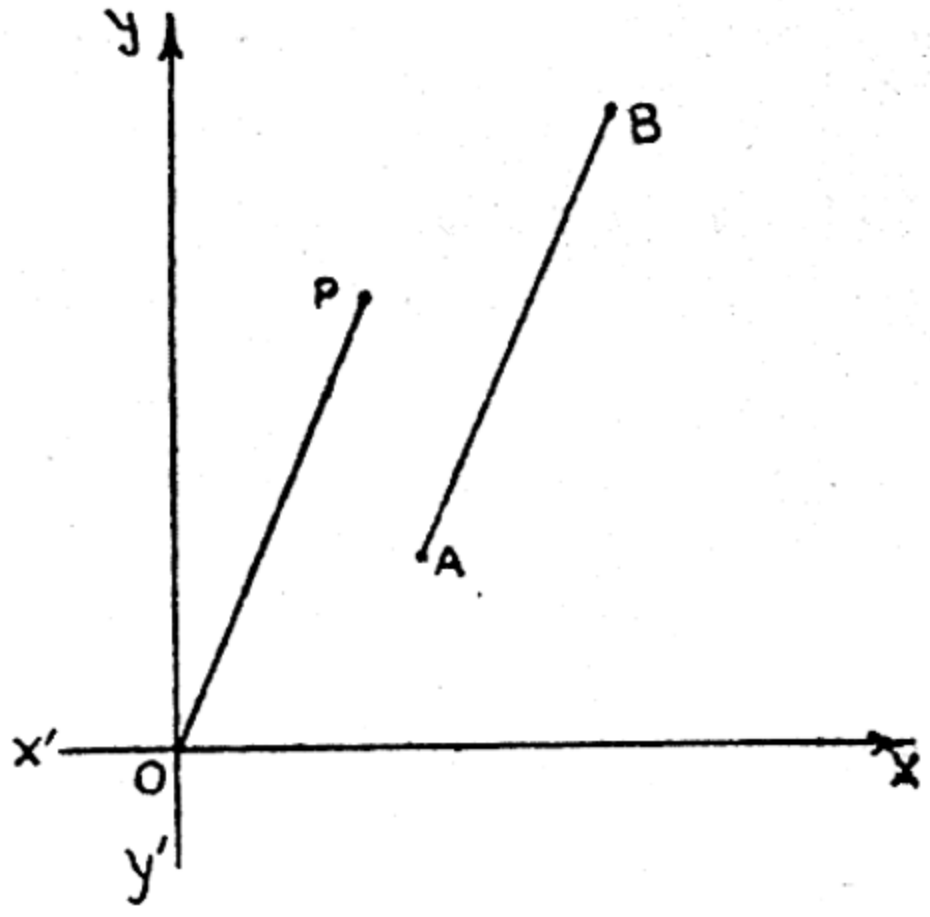


Fig. 1

With each complex number $z=(x,y)$, we associate that point of the rectangular cartesian which has the coordinates x and y, and conversely, with each point having the coordinates x and y, we associate the complex number $z=(x,\ y)$. In the Argand Diagram the x-axis and y-axis are respectively called the *real axis* and *imaginary axis*.

The number z can also be thought of as a vector from the origin to the point $(x,\ y)$; also as any vector obtained by translating that vector in the plane. The complex number z is often referred to as the vector z.

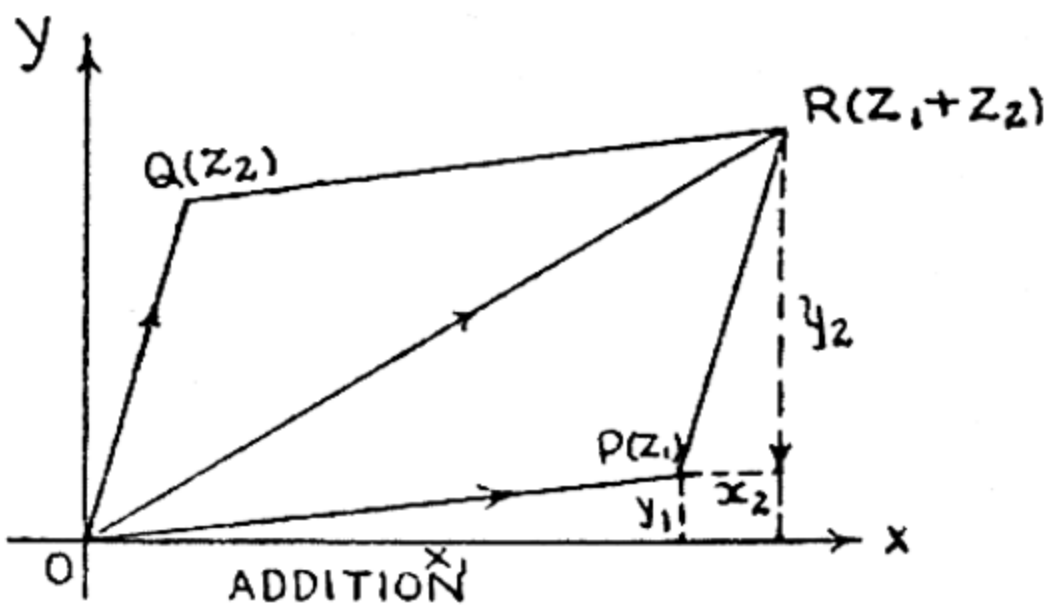


Fig. 2

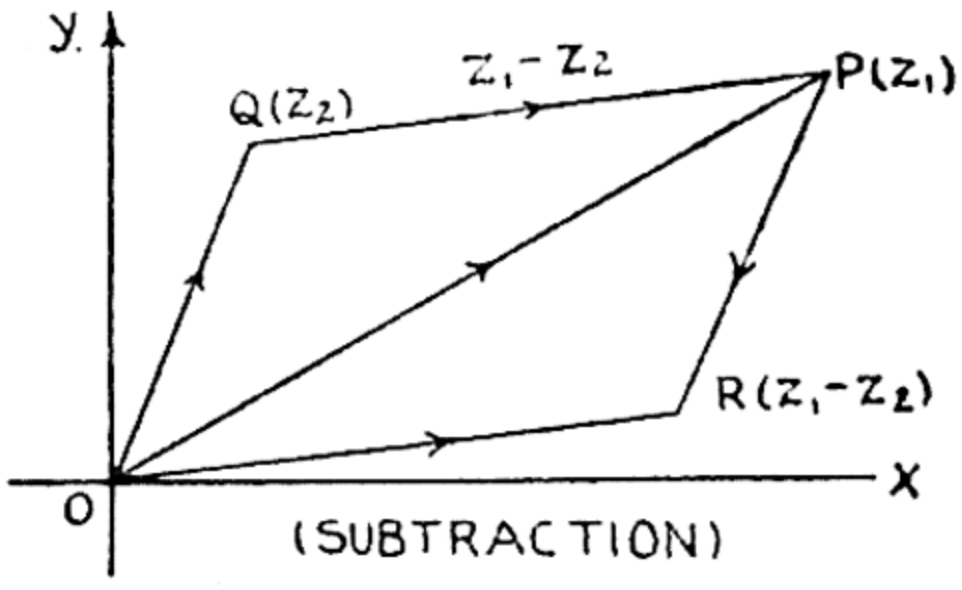


Fig. 3

According to the definition of the sum of two complex numbers $z_1=(x_1,y_1)$ and $z_2=(x_2,\ y_2)$, z_1+z_2 corresponds to the point (x_1+x_2,y_1+y_2). It also corresponds to a vector with those coordinates as its components. Hence, z_1+z_2 is represented by the vector sum

of vector z_1 and z_2 as shown in Fig. 2. The difference z_1-z_2 is respresented by a vector from the point z_2 to the point z_1 (Fig. 3).

Note. The complex number z is also called the **affix** of the point (x, y) which represents it.

1.5. The Polar Form. If (r,θ) be the polar coordinates of the point P (x,y) (Fig. 1 § 1.4),

then $x=r\cos\theta$, $y=r\sin\theta$, ...(1)

and $r=\sqrt{x^2+y^2}$, $\theta=\tan^{-1}\frac{y}{x}$.(2)

The relations (1) are correct in sign, if r is always taken as positive and the positive direction of the angle is chosen to be counter-clockwise.

Hence, every complex number $z=x+iy$ can be written in the form $z=r\ (\cos\theta+i\sin\theta)$, where $r=\sqrt{x^2+y^2}$ is the modulus ; θ is called the **amplitude** (or **argument**] of z, and is written amp z (or arg z).

The amplitude θ is a many-valued function of x and y, the values differing by 2π. The value of θ satisfying the inequality $-\pi<\theta\leqslant\pi$ is called the **principal value** of amp z.

1.6. Products and Quotients.

I. Let $z_1=x_1+iy_1=r_1\ (\cos\theta_1+i\sin\theta_1)$,

and $z_2=x_2+iy_2=r_2\ (\cos\theta_2+i\sin\theta_2)$.

By the definition of multiplication, we have

$$z_1z_2=x_1x_2-y_1y_2+i\ (x_1y_2+x_2y_1)$$

$$=r_1r_2\ \{(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+i\ (\cos\theta_1\sin\theta_2+\cos\theta_2\sin\theta_1)\}$$

$$=r_1r_2\ \{\cos\ (\theta_1+\theta_2)+i\sin\ (\theta_1+\theta_2)\}.$$

By continued application of this equation, we have

$$z_1z_2z_3\ldots z_n=r_1r_2r_3\ldots r_n\times$$
$$\{\cos\ (\theta_1+\theta_2+\theta_3+\ldots\ldots+\theta_n)$$
$$+i\sin(\theta_1+\theta_2+\theta_3+\ldots+\theta_n)\},$$
$$\ldots\ldots(1)$$

where $z_3, z_4,\ldots, z_n$ are similar numbers.

Geometrically, the length of the vector $z_1z_2\ldots z_n$ is equal to the product of the lengths of the vectors $z_1, z_2,\ldots z_n$, *i. e.*, $|z_1z_2\ldots z_n|=|z_1|.|z_2|\ldots|z_n|$, and the amplitude of $z_1z_2\ldots z_n$ is equal to the sum of the amplitudes of $z_1,z_2,\ldots z_n$.

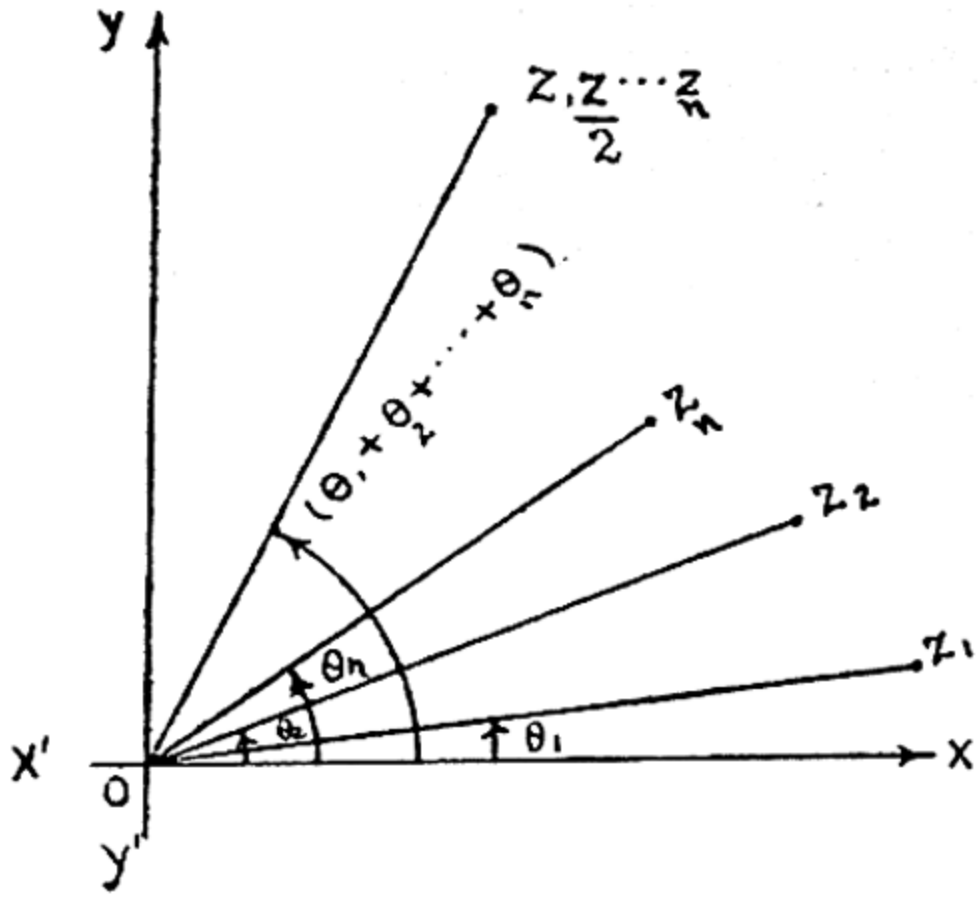


Fig. 5

II. Since division is the inverse of multiplication, we have

$\frac{z_1}{z_2}=\frac{r_1}{r_2}\{\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\}$, for this last is the number which when multiplied by z_2 produces z_1.

Hence, *the modulus and amplitude of a quotient are respectively the quotient of the moduli and difference of the amplitudes of the numerator and the denominator.*

In particular, $\frac{1}{z}=\frac{1}{r}\{\cos(-\theta)+i\sin(-\theta)\}$

$$=\frac{1}{r}(\cos\theta-i\sin\theta),\qquad\ldots\ldots(2).$$

i.e. $\left|\frac{1}{z}\right|=\frac{1}{|z|}$, and $\text{amp}\left(\frac{1}{z}\right)=-\text{amp } z.$

In equation (1), if we put $z_1=z_2=\ldots=z_n=z=r(\cos\theta+i\sin\theta)$, then $z^n=r^n(\cos n\theta+i\sin n\theta),\qquad\ldots\ldots(3)$

where n is any positive integer.

Similarly, from equation (2), it follows that

$$z^{-n}=r^{-n}(\cos n\theta-i\sin n\theta).\qquad\ldots\ldots(4)$$

If $n=\frac{p}{q}$, where p and q are integers and q is positive, then

since $\left(\cos\frac{\theta}{q}+i\sin\frac{0}{q}\right)^q=\cos\theta+i\sin\theta$,

$\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}$ in a value of $(\cos\theta+i\sin\theta)^{\frac{1}{q}}$.

Also, since p is an integer,

$$\left(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}\right)^p=\cos\frac{p}{q}0+i\sin\frac{p}{q}0$$

$$\therefore\quad \cos\frac{p}{q}\theta+i\sin\frac{p}{q}\theta \text{ is a value of } (\cos 0+i\sin 0)^{\frac{p}{q}} \quad \ldots\ldots(5)$$

The equations (3), (4) and (5) are known as De Moivre's Theorem.

Note 1. It should be observed that if 0_1 and 0_2 are the principal values of amp z_1 and amp z_2, then 0_1+0_2 is not necessarily the principal value of amp (z_1+z_2). Similarly, 0_1-0_2 is not necessarily the principal value of amp $\left(\frac{z_1}{z_2}\right)$.

Note 2. $\cos\theta+i\sin\theta$ is written briefly as cis θ, so that De Moivre's Theorem can be written as $(\text{cis}\,\theta)^n=\text{cis}\,n\theta$.

1.61. Geometrical Interpretation of Multiplication by i. When a complex number z is multiplied by i, the resulting vector iz is the one obtained by rotating the vector z through a right angle in the positive direction, without changing the length of the vector, since

$$iz=\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right).\quad r(\cos 0+i\sin 0)$$

$$=r\left\{\cos\left(0+\frac{\pi}{2}\right)+i\sin\left(0+\frac{\pi}{2}\right)\right\}.$$

1.7. Complex Conjugates. The complex conjugate, or simply the **conjugate,** of the complex number $z=x-iy$ is the number $\bar{z}=x-iy$.

Geometrically, the conjugate of z is the **reflection** (or **image**) of z in the real axis.

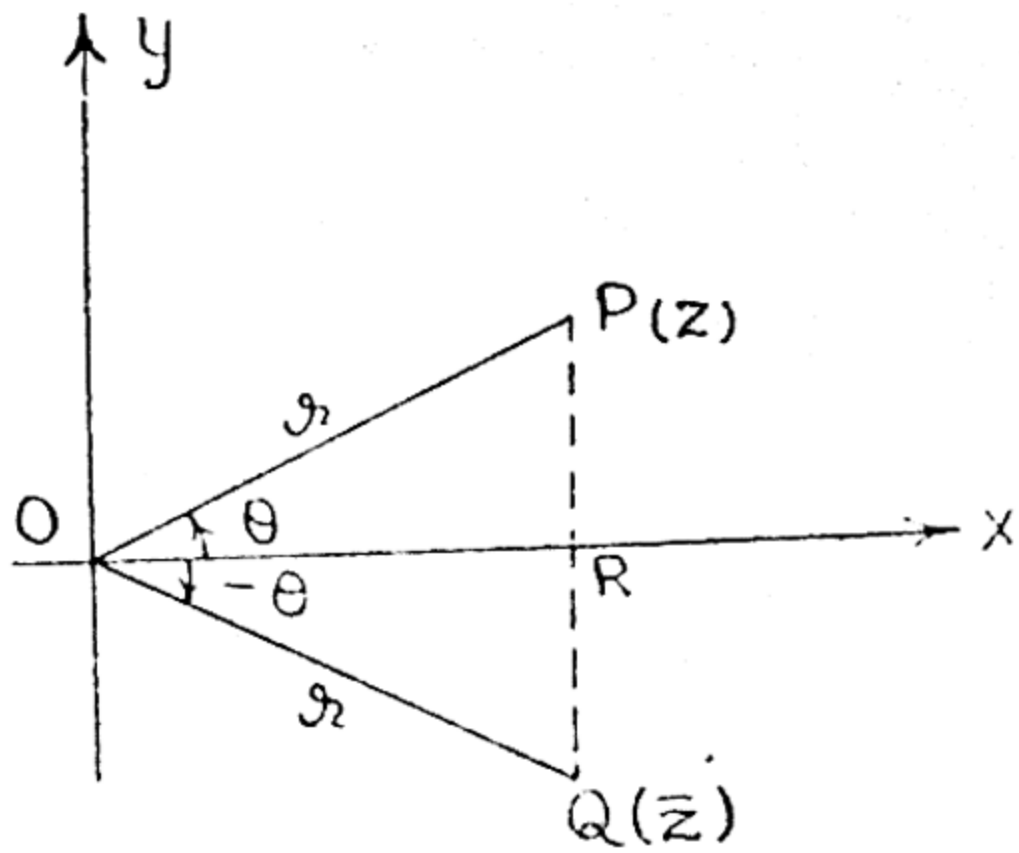


Fig 5

Also, their product is x^2+y^2, and if (r,θ) are the polar coordinates of the point z, those of the point $\bar{z}$ are $(r,-\theta)$. Hence, $|z|^2=z\bar{z}$(1)

$|z|=r=|\bar{z}|$, ...(2)

and amp $z=-$amp $\bar{z}$.(3)

The proofs of many theorems regarding complex numbers can often be greatly simplified by the use of conjugate complex numbers, if we remember the easily proved formulae.

$|z|^2=z\bar{z}$, $z+\bar{z}=2$ Re (z), $z-\bar{z}=2i$ Im (z).

e.g. $|z_1z_2|^2=z_1z_2.\bar{z}_1\bar{z}_2=z_1\bar{z}_1.\ z_2\bar{z}_2=|z_1|^2|z_2|^2$,

and so $|z_1z_2|=|z_1|.|z_2|$. (already proved).

1.8. Unimodular Numbers. From the equation

$a=|a|u,\ a\neq o$;(1)

it follows that $u\bar{u}=1,\ |u|=1$(2)

Numbers u which satisfy (2) are called **unimodular** (*i.e.* of modulus unity). On taking $u=x+iy$, $\bar{u}=x-iy$, ...(3)

we find that $x^2+y^2=1$, ...(4)

so that unimodular numbers may also be defined by equation (4).

The unimodular number -1 satisfies the equation

$$-1=\frac{i}{-i}=\frac{i}{\bar{i}} \qquad \text{.....(5)}$$

Further, every unimodular number u satisfies the equation

$$u(1+\bar{u})=u+u\bar{u}=1+u.$$

Hence, if $u \neq -1$, $u=\dfrac{1+u}{1+\bar{u}}=\dfrac{a}{\bar{a}}$, ...(6)

where $a=1+u\neq 0$.

Thus, every unimodular number can be represented as the quotient of two complex conjugates, and vice versa. Hence, if $z\neq a$ $\dfrac{z-a}{\bar{z}-\bar{a}}$ is unimodular. Further, if $|z|=1$, then $\dfrac{-1}{z}$ is unimodular

and so is $$w=-\frac{1}{z}\cdot\frac{z-a}{\bar{z}-\bar{a}}=\frac{a-z}{1-\bar{a}z}, \qquad \text{.....(7)}$$

where $|a|\neq 1$. Solving this for z, we have $z=-\dfrac{a-w}{1-\bar{a}w}$, ...(8)

which shows that each of the equations (7) and (8) may be obtained from the other by interchanging z and w.

From (7) we also have $|w|^2=\dfrac{(a-z)(\bar{a}-\bar{z})}{(1-\bar{a}z)(1-az)}$,

and $$1-|w|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}. \qquad \text{.....(9)}$$

Hence, if $|a|<1$, $(1-|z|^2)$ and $(1-|w|^2)$ both have the same sign. It follows, therefore, from (7) and (8) that to every number z whose modulus $\leqslant 1$ corresponds a number w having the same property.

We shall see later (Ex. 3, §3.47), that the transformation (7) maps the unit circle $|z|\leqslant 1$ on the unit circle $|w|\leqslant 1$.

1.9. SOME IMPORTANT THEOREMS.

Theorem I. **The modulus of the sum of any number of complex quantities is never greater than the sum of their moduli.**

Geometrical Proof. Let P and Q be the points z_1 and z_2 respectively and R the point z_1+z_2 [see Fig. 2, § 1·4]. Obviously, $OR<OP+OQ$; $\therefore$ $|z_1+z_2|<|z_1|+|z_2|$. The sign of equality will occur only, if P, Q, R are collinear. By induction, the theorem can be proved for more than two numbers.

Algebraical Proof. Let $z_1=r_1\ (\cos\theta_1+i\sin\theta_1)$, and $z_2=r_2\ (\cos\theta_2+i\sin\theta_1)$,

then
$$\begin{aligned}|z_1+z_2| &= |(r_1\cos\theta_1+r_2\cos\theta_2)+i(r_1\sin\theta_1+r_2\sin\theta_2)]\\ &=\sqrt{[(r_1\cos\theta_1+r_2\cos\theta_2)^2+(r_1\sin\theta_1+r_2\sin\theta_2)^2]}\\ &=\sqrt{[r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)]}\\ &\leqslant \sqrt{(r_1^2+r_2^2+2r_1r_2)},\quad [\because \cos(\theta_1-\theta_2)\leqslant 1]\\ \text{or}\quad &\leqslant r_1+r_2\quad [=|z_1|+|z_2|];\end{aligned}$$
$$\therefore\quad |z_1+z_2|\leqslant |z_1|+|z_2|.$$

The equality will hold only when $\cos(\theta_1-\theta_2)=1$,
i. e. when $\theta_1-\theta_2=n\pi$, n being an integer.

Aliter.
$$\begin{aligned}|z_1+z_2|^2&=(z_1+z_2)(\bar{z}_1+\bar{z}_2)=z_1\bar{z}+z_1\bar{z}_2+\bar{z}_1z_2+z_2\bar{z}_2\\ &=|z_1|^2+2\,\mathrm{Re}(\bar{z}_1z_2)+|z_2|^2\\ &\leqslant |z_1|^2+2|z_1z_2|+|z_2|^2\\ &\quad [\because\ \mathrm{Re}(z)\leqslant |z|,\ i.\,e.\ \mathrm{Re}(\bar{z}_1z_2)\leqslant |\bar{z}_1z_2|\\ &\qquad =|z_1z_2|.]\\ &=(|z_1|+|z_2|)^2;\end{aligned}$$
$$\therefore\quad |z_1+z_2|\leqslant |z_1|+|z_2|.$$

Theorem II. **Modulus of the sum or difference of two complex quantities is never less than the difference of their moduli.**

Geometrical Proof. From Fig. 2, § 1.4, OR+PR>OP, or OR>OP−PR ;

$$\therefore\quad |z_1+z_2|\ >\ |z_1|-|z_2|.$$

The sign of equality holds only when z_1, z_2 are collinear with the origin. Similarly, in Fig. 3, § 1.4, taking the triangle with sides of length $|z_1|$, $|z_2|$ and $|z_1-z_2|$, we have $|z_1-z_2|\geqslant |z_1|-|z_2|$, since no side of a triangle is less than the difference of the other two.

Algebraical Proof. Let $z_1=r_1\ (\cos\theta_1+i\sin\theta_1)$,
and $z_2=r_2\ (\cos\theta_2+i\sin\theta_2)$,

then
$$\begin{aligned}|z_1+z_2| &=\sqrt{[r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)]}\\ &\geqslant |r_1-r_2|,\qquad [\because\ \cos(\theta_1-\theta_2)\geqslant -1.]\\ \text{or}\quad &\geqslant |\ |z_1|-|z_2|\ |.\end{aligned}$$

The sign of equality holds only when $\cos(\theta_1-\theta_2)=-1$,
i.e., when $\theta_1-\theta_2=(2n+1)\pi$, n being an integer.

Similarly, $|z_1-z_2|^2=r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)$
$\geqslant(r_1-r_2)^2$; $[\because \cos(\theta_1-\theta_2)\leqslant 1.]$

$\therefore \quad |z_1-z_2|\geqslant||z_1|-|z_2||$.

The sign of equality holds only when $\theta_1-\theta_2=2n\pi$, n being an integer.

Aliter. $|z_1\pm z_2|^2=(z_1\pm z_2)(\bar{z}_1\pm\bar{z}_2)=z_1\bar{z}_1\pm(z_1\bar{z}_2\pm z_2\bar{z}_1)+z_2\bar{z}_2$
$=|z_1|^2\pm 2\,\mathrm{Re}\,(z_1\bar{z}_2)+|z_2|^2$
$\geqslant|z_1|^2-2|z_1z_2|+|z_2|^2$
$=(|z_1|-|z_2|)^2$;

$\left[\begin{array}{l}\because \quad \mathrm{Re}\,(z_1\bar{z}_2)\leqslant|z_1\bar{z}_2|=|z_1z_2|,\\ \therefore \quad \pm\,\mathrm{Re}\,(z_1\bar{z}_2)\geqslant-|z_1z_2|.\end{array}\right.$

$\therefore \quad |z_1\pm z_2|\geqslant||z_1|-|z_2||$.

Theorem III. **Prove that**

$$|\mathbf{z}_1+\mathbf{z}_2|^2+|\mathbf{z}_1-\mathbf{z}_2|^2=2|\mathbf{z}_1|^2+2|\mathbf{z}_2|^2.$$

Proof. Let $z_1=r_1(\cos\theta_1+i\sin\theta_2)$, and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, then

$|z_1+z_2|^2+|z_1-z_2|^2=[r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)]$
$+[r_1^2+r_1^2-2r_1r_2\cos(\theta_1-\theta_2)]$
$=2(r_2^2+r_2^2)=2|z_1|^2+2|z_2|^2$.

Aliter. $|z_1+z_2|^2+|z_1-z_2|^2$
$=(z_1+z_2)(\bar{z}_1+\bar{z}_2)+(z_1-z_2)(\bar{z}_1-\bar{z}_2)$
$=2(z_1\bar{z}_1-z_1\bar{z}_2)=2|z_1|^2+2|z_2|^2$.

Ex. 1. *Find the equation to a straight line joining two given points z_1, z_2 in the Argand plane.*

Let z be any point on the straight line joining z_1,z_2, then

$$\text{amp}\left(\frac{z-z_1}{z-z_2}\right)=\pi, \text{ or } 0,$$

according as z lies inside the line joining z_1, z_2 or outside it.

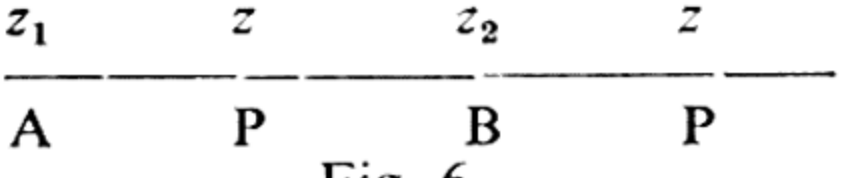


Fig. 6

In both the cases, $\text{Im}\left(\frac{z-z_1}{z-z_2}\right)=0$, *i.e.*, $\frac{z-z_1}{z-z_2}=\frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}$,

or $z(\bar{z}_1-\bar{z}_2)-\bar{z}(z_1-z_2)+(z_1\bar{z}_2-z_2\bar{z}_1)=0$, ...(1)

which is the required equation.

If $z_1=a_1+i\,b_1$, $z_2=a_2+i\,b_2$, then $z_1-z_2=(a_1-a_2)+i(b_1-b_2)=i\alpha$, say, $\bar{z}_1-\bar{z}_2=(a_1-a_2)-i\,(b_1-b_2)=-i\alpha$, $z_1\bar{z}_2-z_2\bar{z}_1=-2i\,(a_1b_2-a_2b_1)=-ic$, say; where c is real. The eqution (1) can thus be written

$$-i\bar{\alpha}z-i\alpha\bar{z}-ic=0,$$

or $\bar{\alpha}z+\alpha\bar{z}+c=0$,.....(2)

where c is real.

Ex. 2. *Show that, if* $amp\left(\frac{z_2-z_3}{z_1-z_3}\right) = amp\left(\frac{z_2-z_4}{z_1-z_4}\right)$, *the points* z_3 *and* z_4 *are on the same side of the line joining* z_1 *and* z_2, *and* z_1, z_2, z_3, z_4 *are concyclic.*

Let the points P_1, P_2, P_3 and P_4 represent respectively z_1, z_2, z_3 and z_4 (Fig. 7 (i)] ; then

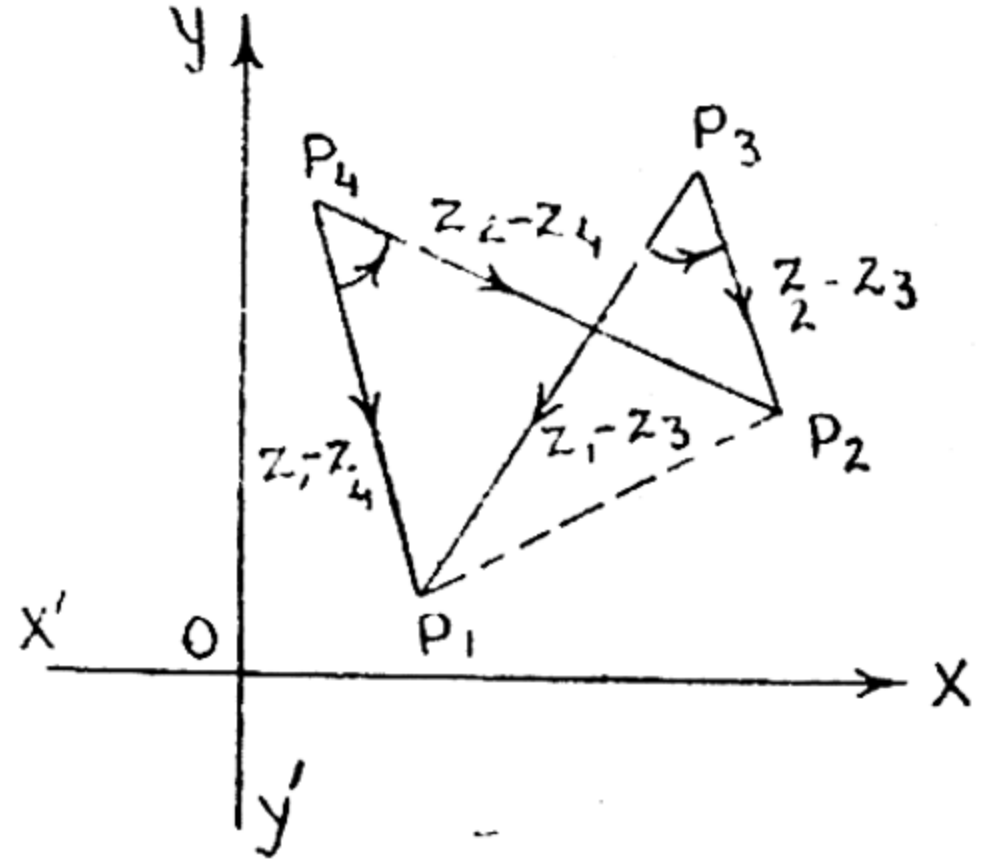


Fig. 7. (i)

$\text{amp}\left(\frac{z_2-z_3}{z_1-z_3}\right) = \angle P_1P_3P_2$, and $\text{amp}\left(\frac{z_2-z_4}{z_1-z_4}\right) = \angle P_1P_4P_2$.

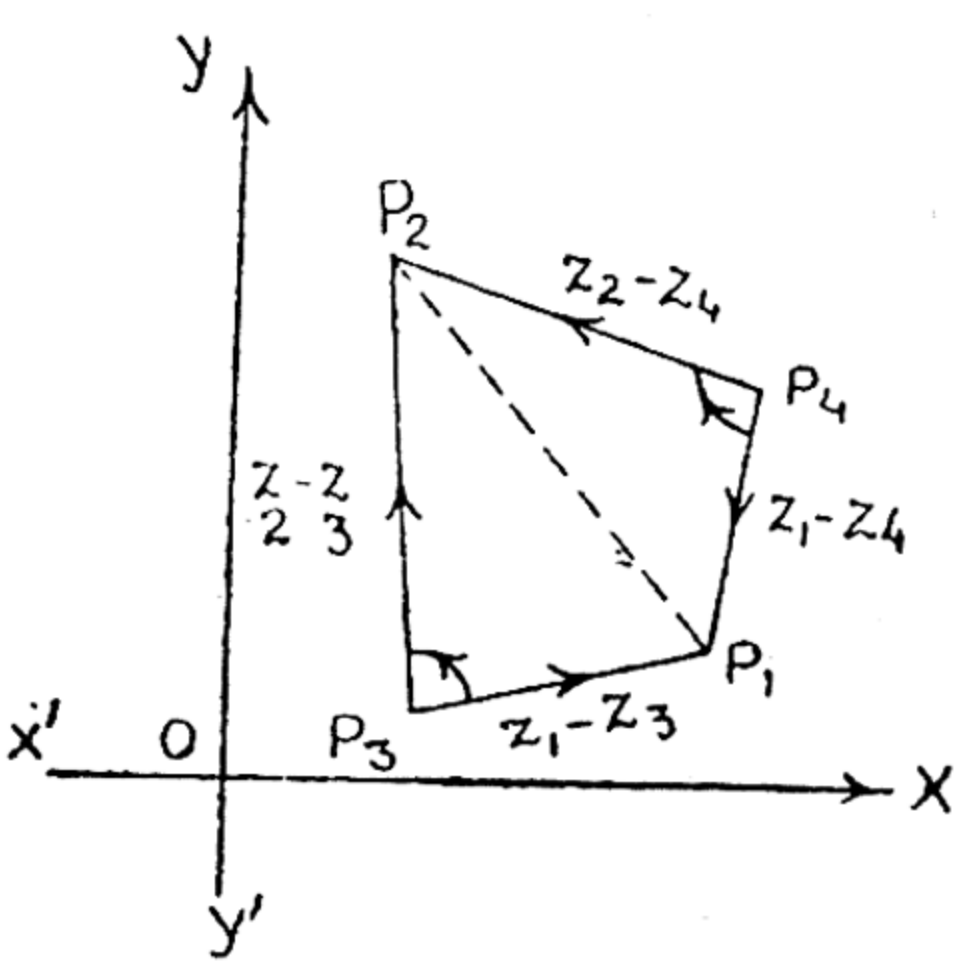


Fig. 7. (ii)

The points P_3 and P_4 must be on the same side of P_1P_2, otherwise the angles $\angle P_1P_3P_2$ and $\angle P_1P_4P_2$ will have opposite signs [Fig. 7(ii)].

Also, in Fig. (i), angles in the same segment are equal ;

[$\because$ $\angle P_1P_3P_2 = \angle P_1P_4P_2$, given]

$\therefore$ the points P_1, P_2, P_3, P_4 are concyclic.

Ex. 3. *Prove that* $amp(z) - amp(-z) = \pm\pi$, *according as* $amp(z)$ *is possitive or negative.*

Let $\text{amp}(z)=0$, then $-\pi<\theta\leqslant\pi$.

If $0<\theta\leqslant\pi$, $\text{amp}(-z)=-(\pi-\theta)$;

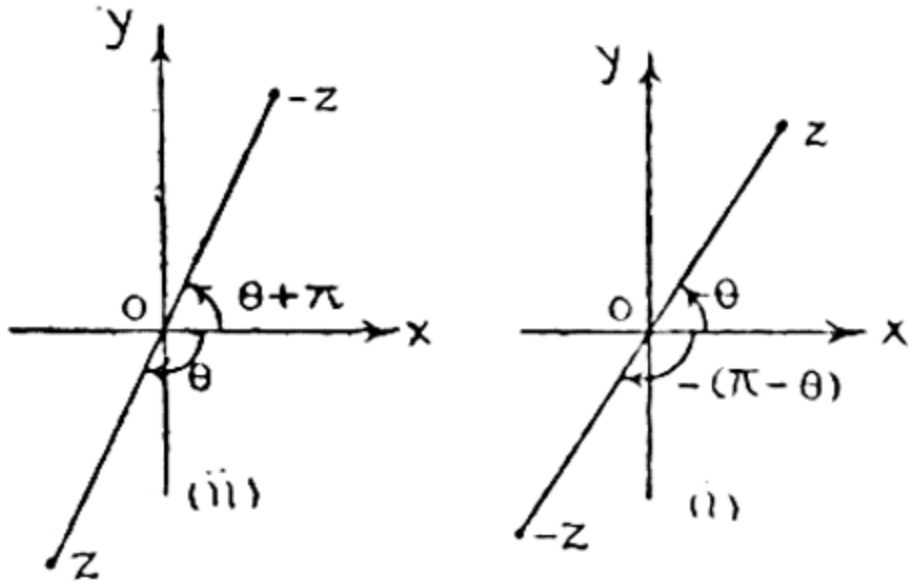


Fig. 8.

$\therefore$ $\text{amp}(z) - \text{amp}(-z) = \pi$.

If $0>\theta>-\pi$, $\text{amp}(-z)=0+\pi$;

$\therefore$ $\text{amp}(z) - \text{amp}(-z) = -\pi$.

Hence, $\text{amp}(z) - \text{amp}(-z) = \pm\pi$, according as $\text{amp}(z)$ is positive or negative.

Ex. 4. *Show that, if* $amp\left\{\dfrac{(z_2-z_3)(z_1-z_4)}{(z_1-z_3)(z_2-z_4)}\right\}=\pi$, *then* z_3 *and* z_4 *are on the opposite sides of the line joining* z_1 *and* z_2, *and* z_1, z_2, z_3; z_4 *are concyclic.*

Now $\text{amp}\left\{\dfrac{(z_2-z_3)(z_1-z_4)}{(z_1-z_3)(z_2-z_4)}\right\}=\pi$,

or $\text{amp}\left(\dfrac{z_2-z_3}{z_1-z_3}\right)-\text{amp}\left(\dfrac{z_2-z_4}{z_1-z_4}\right)=\pi$.(1)

If the points z_2, z_4 are on the same side of the line joining z_1 and z_2 ; then both the amplitudes have the same sign and equation (1) is impossible (Ex. 2).

Hence, the points z_3, z_4 must be on opposite sides of the line joining z_1 and z_2. Also, in this case the amplitudes have opposite signs (Ex. 2) ;

$$\therefore \quad \text{if amp } \left(\frac{z_2-z_3}{z_1-z_3}\right) = \theta_1, \text{ then amp } \left(\frac{z_2-z_4}{z_1-z_4}\right) = -\theta_2,$$

where θ_1 and θ_2 have the same sign. Hence, $\theta_1+\theta_2=\pi$, which shows that the four points z_1, z_2, z_3. z_4 are concyclic.

Note. The angle θ_1 is positive [see Fig. 7 (ii)].

Ex. 5. *Prove that the two triangles whose vertices are the points* a_1, a_2, a_3 *and* b_1, b_2, b_3 *respectively, are directly similar if, and only if*

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0.$$

If A,B, C and P, Q, R be the corresponding vertices of the two triangles. then condition for similarity gives

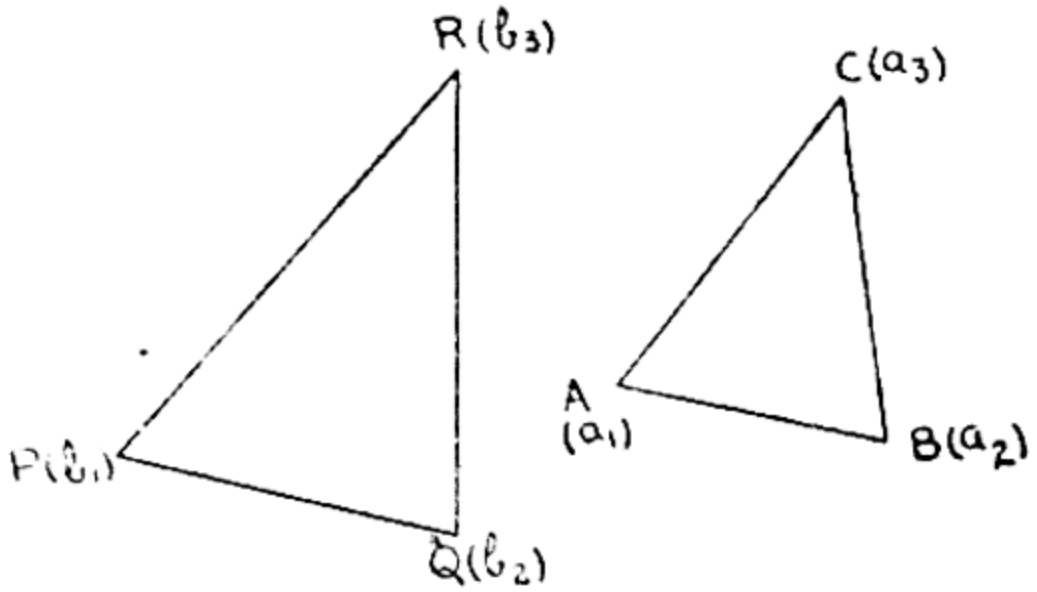


Fig. 9

$$\frac{\overrightarrow{AB}}{\overrightarrow{AC}} = \frac{\overrightarrow{PQ}}{\overrightarrow{PR}},$$

or $$\frac{a_2-a_1}{a_3-a_1} = \frac{b_2-b_1}{b_3-b_1};$$

or $$a_1(b_2-b_3)+a_2(b_3-b_1)+a_3(b_1-b_2)=0,$$

or $$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0.$$

Note. a_1, b_1, etc. are complex numbers.

Ex. 6. *Prove that*

$$|\alpha+\sqrt{\alpha^2-\beta^2}| + |\alpha-\sqrt{\alpha^2-\beta^2}| = |\alpha+\beta| + |\alpha-\beta|.$$

(Agra 1946, '53)

Now, $|z^2| = |x^2-y^2+2ixy| = \sqrt{[(x^2-y^2)^2+4x^2y^2]}$

$=x^2+y^2=|z|^2$.(1)

Also, $|z_1z_2| = |z_1| \cdot |z_2|$.(2)

$\therefore \quad \{|\alpha+\sqrt{\alpha^2-\beta^2}| + |\alpha-\sqrt{\alpha^2-\beta^2}|\}^2$

$= |\alpha+\sqrt{\alpha^2-\beta^2}|^2 + |\alpha-\sqrt{\alpha^2-\beta^2}|^2$

$+2|\alpha+\sqrt{\alpha^2-\beta^2}|\,|\alpha-\sqrt{\alpha^2-\beta^2}|$

$=2|\alpha|^2+2|\sqrt{\alpha^2-\beta^2}|^2+2|\alpha^2-(\alpha^2-\beta^2)|$ [see Th. III. § 1.9]

$=2|\alpha|^2+2|\alpha^2-\beta^2|+2|\beta|^2$, by (1)

$= |\alpha+\beta|^2+|\alpha-\beta|^2+2|\alpha+\beta|\,|\alpha-\beta|$

[by Th. III, §1.9 and (2) above]

$=\{|\alpha+\beta|+|\alpha-\beta|\}^2$.

Hence, $|\alpha+\sqrt{\alpha^2-\beta^2}|+|\alpha-\sqrt{\alpha^2-\beta^2}| = |\alpha+\beta|+|\alpha-\beta|$.

Exercise 1

1. Prove that the equation of a circle in the Argand plane is
$$z\bar{z}+b\bar{z}+\bar{b}z+c=0,$$
where b,c are constants, c being real.
[Hint. Take the equation as $|z-z_1|=r$
which gives $(z-z_1)(\bar{z}-\bar{z}_1)=r^2$, etc.]

2. Find the equation of the circle having the line joining z_1 and z_2 as diameter.

3. The two lines joining the points $z=a$, $z=b$ and $z=c$, $z=d$ will be perpendicular, if amp $\left(\frac{a-b}{c-d}\right)=\pm\frac{\pi}{2}$, *i.e.*, if $\frac{a-b}{c-d}$ is purely imaginary. What is the condition that the lines should be parallel ?

4. Find the centroid, circumcentre and the orthocentre of the triangle whose vertices are z_1, z_2, z_3.

5. Show that the four points are z_1, z_2, z_3 and z_4 are concyclic, if and only if, the expression $\frac{(z_2-z_3)(z_1-z_4)}{(z_1-z_3)(z_2-z_4)}$ is real.

6. Prove that the curves $\left|\frac{z-1}{z+1}\right|=$ constant and amp $\left(\frac{z-1}{z+1}\right)$ $=$constant are orthogonal circles.

7. Find all the values of z which satisfy the equation $(z+1)^5+z^5=0$ and show that their representative points all lie on a straight line parallel to the imaginary axis.

8. If a, b are complex numbers, find numbers z_1, z_2 so that the points z_1, z_2 and a,b may be opposite corners of a square.

9. If $|z_1| = |z_2|$, and amp z_1 + amp $z_2 = 0$, show that z_1 and z_2 are conjugate numbers.

10. A, B, C and D are the points z_1, z_2, z_3 and z_4 respectively. Show that, if $z_1z_2 + z_3z_4 = 0$ and $z_1 + z_2 = 0$, then A,B,C and D are concyclic and the triangles AOC and DOC are similar.

11. If z_1, z_2, z_3 are the vertices of an isosceles triangle, right-angled at the vertex z_2, prove that $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

[**Hint.** $\dfrac{z_2 - z_1}{z_3 - z_2} = \pm i$. Square both sides.

12. If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$ the affixes z_1, z_2, z_3 form an equilateral triangle in the unit circle $|z| = 1$.

13. If z_1, z_2, z_3 are the vertices of an equilateral triangle, show that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

14. If $(z_1 - z_2)(z'_1 - z'_2) = (z_2 - z_3)(z'_2 - z'_3) = (z_3 - z_1)(z'_3 - z'_1)$, show that the triangles whose vertices are z_1, z_2, z_3 and z'_1 z'_2, z'_3 are equilateral.

[**Hint.** $\dfrac{z_1 - z_2}{z'_2 - z'_3} = \dfrac{z_2 - z_3}{z'_1 - z'_2}$ (given) $= \dfrac{z_1 - z_3}{z'_1 - z'_3}$.(1)

Also $(z_2 - z_3)(z'_2 - z'_3) = (z_3 - z_1)(z'_3 - z'_1)$(2) (given)
Multiply (1) and (2) and use the result of the previous example.]

15. Prove that the area of a triangle whose vertices are the points z_1, z_2, z_3 on the Argand Diagram can be written in either of the following forms :

(i) Absolute value of $\frac{1}{2} |z_2 - z_3|^2 \operatorname{Im}\left\{\dfrac{z_3 - z_1}{z_3 - z_2}\right\}$.

(ii) $\dfrac{1}{4i} \Sigma [(z_2 - z_3) |z_1|^2 / z_1]$

[**Hint.** Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ etc., then

the area $= \frac{1}{2} \Sigma r_1 r_2 \sin (\theta_2 - \theta_1)$

$$= \frac{1}{4i} \Sigma r_1 r_2 \left\{ e^{i(\theta_2 - \theta_1)} - e^{-i(\theta_2 - \theta_1)} \right\}$$

$$=\frac{1}{4i}\Sigma r_1 e^{-i\theta_1}\left(r_2 e^{i\theta_2}-r_3 e^{i\theta_3}\right)=\frac{1}{4i}\Sigma\{(z_2-z_3)\bar{z}_1\}$$

$$=\frac{1}{4i}\Sigma\frac{(z_2-z_3)\mid z_1\mid^2}{z_1}.\Big]$$

ANSWERS

2. $2z\bar{z}-(\bar{z}_1+\bar{z}_2)z-(z_1+z_2)\bar{z}+(z_1\bar{z}_2+z_2\bar{z}_1)=0.$

4. $\frac{1}{3}(z_1+z_2+z_3)\,;\frac{\Sigma z_1\bar{z}_1(z_2-z_3)}{\Sigma(\bar{z}_2z_3-z_2\bar{z}_3)}\,;\ \frac{\Sigma z_1{}^2(\bar{z}_2-\bar{z}_3)+\Sigma z_1\bar{z}_1(z_2-z_3)}{\Sigma(z_2\bar{z}_3-z_3\bar{z}_2)}.$

7. $-\frac{1}{2}+\frac{i}{2}\tan\frac{k0}{2}(k=0,\ 1,2,3,4).$

8. $z_1=\frac{1}{2}(a+b)-\frac{1}{2}\,i\,(a-b),\ z_2=\frac{1}{2}(a+b)+\frac{1}{2}\,i\,(a-b).$

CHAPTER II

ANALYTIC FUNCTIONS

2.1. **Constants, Variables.** The same distinction is made between complex constants and complex variables as between real constants and real variables. If a complex number takes up only one value in any discussion, it is called a *constant*. On the other hand, if a complex number assumes different complex values, it is called a *variable*. A complex variable is written in the form $x+iy$, where x and y are real variables.

2.11. **Regions in the Complex Plane.** By a *neighbourhood* of a point a in the Argand plane, we mean the set of all points z such that $|z-a|<\varepsilon$, where ε is an arbitrarily small positive number; ε is called the radius of this neighbourhood. This circular region includes the point a itself and excludes the points on the circle.

In referring to points of neighbourhood of a exclusive of the point a itself, we shall speak of the region as a *deleted neighbourhood* of a.

A point a is said to be a **limiting point** (or **limit point**) of a set of points S in the Argand plane if every neighbourhood of a contains a point of S distinct from a. The point a may or may not be a point of the set. To illustrate, each point on the circle $|z|=r$ is a limit point of the set $|z|<r$. These points do not belong to the set. But each point inside the circle is also a limit point that does belong to the set. A set of points S is said to be *closed* if every limit point of the set belongs to the set.

There are two types of limit points, *interior* (or *inner*) points and *boundary* points. A limit point a of S is an interior point if all points within a neighbourhood of a belong to S. A limit point which is not an interior point is a boundary point. A set which consists entirely of interior points is said to be an *open set* or **open**

two dimensional region (or **domain**). By the word **region** we mean a *connected* (or *connex*) region, one such that every pair of its points can be joined by a continuous curve which consists only of points of the region. An open connected region is called an **open region** (or **domain**). If the region includes all its limit points, such as the region $|z| \leqslant r$, it is called a *closed* region.

A region is *bounded*, if there exists a positive finite number r for which $|z| \leqslant r$ is satisfied for all points z of the region. If there exists no such number r, the region is said to be *unbounded*.

Note. Unless the contrary is stated, all points, in particular the limit points, will represent finite complex numbers. Hence, a closed region will mean a bounded closed region.

2.12. Jordan arcs, Differentiable arcs, Regular arcs.

The set of points z determined by the equation $z=x(t)+iy(t)$, where $x(t)$ and $y(t)$ are continuous functions of the real variable t in the closed interval (t_o, t_n), is called a continuous arc. If a continuous arc has no *multiple point*, *i.e.*, for no two different values t_r, t_s of t in the above interval $x(t_r)=x(t_s)$, $y(t_r)=y(t_s)$, it is called a **Jordan arc**. A simple example of a Jordan curve is a non-intersecting polygonal arc consisting of a finite chain of linear segments. A continuous curve which has only one multiple point, *viz.* a double point corresponding to the end values t_o, t_n of t is called a **simple closed Jordan curve**.

For example, if $x(t)=\cos t$, $y(t)=\sin t$, where $0 \leqslant t \leqslant 2\pi$, the set of points $z=x(t)+iy(t)$ determines a circle which is thus a simple closed Jordan curve having the double point at $z=1$ corresponding to $t=0$ and $t=2\pi$.

If the Jordan arc $z=x(t)+iy(t)$, $t_o \leqslant t \leqslant t_n$, possesses tangents at all its points, *i.e.*, $x'(t)$ and $y'(t)$ exist for all t in (t_o, t_n) and do not both vanish at the same time, it is called a **differentiable arc**. If, further, this arc possesses a continuously turning tangent, *i.e.*, the derivatives $x'(t)$ and $y'(t)$ arc continuous, it is called a **regular arc** (or **smooth curve**)

It can be proved that a regular arc is rectifiable.

2.2. Functions of a Comlpex Variable. The complex variable w is said to be a *function* of the complex variable z in a given

open or closed regin R, if for each value of z in this region w has a definite value or set of values.

Here, as in the case of functions of a real variable, the functions may be defined also w.r.t a set of values rather than for all values of z of a given region. Unless otherwise stated, however, it will be understood that z takes all values of a given region. Then, if w is a function of z, we may write $w=u(x,y)+iv\,(x,y)=f(z)$, where u, v are real functions of the two real variables x and y.

If to each value of z there corresponds one value of w, w is said to be a *uniform* or *single-valued* function ; if w takes two or more values for some or all of the values of z, then w is called a *multiple-valued* (or *many-valued*) function of z.

A complex function, *i.e.*, a function of a complex variable may sometimes assume only real values ; *e.g.*, $w=|z|$.

2.21. **Path of Variation.** In the case of real functions, the independent variable can vary by values corresponding to points on the x-axis ; in the case of complex functions, however, the independent variable can vary by values corresponding to the points of any path connecting the initial and final points.

2.3. **Continuity.** The definition is exactly the same as that for functions of a real variable. The function $f(z)$ is *continuous* at the point z_0, if given any $\varepsilon>0$, we can find a number δ such that $|f(z)-f(z_0)|<\varepsilon$, for all points z of the region satisfying $|z-z_0|<\delta$. The number δ depends on ε and also, in general, upon z_0.

The function $f(z)$ is said to be *uniformly continuous* in a region R, if it is possible to find a number $\eta\,(\varepsilon)$ independent of z_0, so that $|f(z)-f(z_0)|<\varepsilon$ for every pair z, z_0 of the region R, when $|z-z_0|<\eta$.

Another definition. A single-valued function $f(z)$ is continuous at a point z_0, if, and only if $f(z_0)$ exists,

$$\text{and} \quad \underset{z \longrightarrow z_0}{\text{Lt}} f(z)=f(z_0).$$

As in the case of real functions, if $f(z)$ is continuous at every point in a closed region R, then it is uniformly continuous there.

The properties of a continuous function may be obtained from the corresponding properties of real functions of two variables, since the function $f(z)=u(x, y)+iv(x, y)$ will be continuous if both u and v are continuous functions of the two variables x and y together.

For example, the function $f(z)=ax^2y+ibxy^2$ is continuous everywhere and so is $f(z)=\cos(xy)+ie^x$ because of the continuity of the real functions x^2y, xy^2, $\cos(xy)$ and e^x.

If $f(z)$ has a definite limit at z_0 different from $f(z_0)$, $f(z)$ is said to have a *removable discontinuity* at z_0, and the function can be made continuous by replacing the value at z_0 by the limit at that point.

To investigate the continuity of a function at infinity, put $z=\frac{1}{z'}$, and test for $z'=0$.

2.4. Differentiability. The definition of the derivative of a complex function $f(z)$ is identical in form with the definition of the derivative of a real function. The general laws of differentiation for real variables can be extended without modification to complex functions. The symbols used are also the same.

Let z be a variable point in the neighbourhood of z_0 in a region R, in which the single-valued function $w=f(z)$ is defined ; then if the limit $\underset{\triangle z \longrightarrow 0}{\text{Lt}} \frac{\triangle w}{\triangle z} = \underset{\triangle z \longrightarrow 0}{\text{Lt}} \frac{f(z_0+\triangle z)-f(z_0)}{\triangle z}$ exists, where $\triangle z \equiv z-z_0$ and $\triangle w=f(z)-f(z_0)=f(z_0+\triangle z)-f(z_0)$, we say that this limit is the *derivative* of $f(z)$ at the point z_0, and $f(z)$ is *differentiable* at z_0. This limit is denoted by $f'(z_0)$.

There is, however, a fundamental difference between this derivative and the derivative of a function of a real variable. If this limit is to exist, the same limiting value must be obtained independently of the path along which z approaches z_0. The value of the limit must, therefore, be independent of the amplitude of $\triangle z$ as $z \longrightarrow z_0$. If the limit depends on amp $(\triangle z)$, the derivative of $f(z)$ at z_0 does not exist.

If $f'(z_0)$ exists, then $f(z)$ is necessarily continuous at $z=z_0$,

for, $[f(z_0+\triangle z)-f(z_0)] \longrightarrow 0$ as $\triangle z \longrightarrow 0$,

if $\underset{\triangle z \longrightarrow 0}{\text{Lt}} \dfrac{f(z_0+\triangle z)-f(z_0)}{\triangle z}$ is to exist.

However, continuity does not imply differentiability as can be seen from the following example :

Let $f(z)=|z|^2$. This function is continuous everywhere because of the continuity of x^2+y^2.

Here $f'(z_0)=\underset{\triangle z \longrightarrow 0}{\text{Lt}} \dfrac{|z_0+\triangle z|^2-|z_0|^2}{\triangle z}$

$$= \underset{\triangle z \longrightarrow 0}{\text{Lt}} \frac{(z_0+\triangle z)\,(\overline{z_0}+\overline{\triangle z})-z_0\overline{z}_o}{\triangle z}$$

$$= \underset{\triangle z \longrightarrow 0}{\text{Lt}} \left(\overline{z}_o+\overline{\triangle z}+z_o\frac{\overline{\triangle z}}{\triangle z}\right).$$

Now, $\overline{\triangle z} \to 0$ as $\triangle z \to$; $\therefore$ at the pt $z_o=0$, the limit is zero, *i. e.*, $f'(0)=0$.

If $z_o \neq 0$, let amp $\triangle z=\varnothing$, then

$$\frac{\overline{\triangle z}}{\triangle z}=\frac{\cos(-\varnothing)+i\sin(-\varnothing)}{\cos\varnothing+i\sin\varnothing}=\cos 2\varnothing-i\sin 2\varnothing,$$

and $\underset{\triangle z \longrightarrow 0}{\text{Lt}} \dfrac{\overline{\triangle z}}{\triangle z}$ does not exist, since the value depends upon amp (z).

Hence, the continuous function $f(z)=|z|^2$ is differentiable at $z_o=0$ and nowhere else.

2.5. **Analytic Functions.** If a single-valued function $f(z)$ has a derivative at a point z_o and at every point in the neighbourhood of z_o, then $z=z_o$ is called a *regular point* of $f(z)$, or the function is said to be *analytic* at $z=z_o$. When it exists, the derivative has, of course, a unique value.

The function $|z|^2$, for example, is not analytic at any point, since its derivative exists only at $z=0$, and therefore, not throughout any neighbourhood.

A function is anylytic in a region of the z-plane if it is analytic at every point in that region. The terms *holomorphic* and *regular* are sometimes used as synonymous with the term analytic.

A point in every deleted neighbourhood of which there are regular points but which is itself not a regular point is called a *singular point* of the given region. For example, the function $f(z)=\frac{1}{z}$ has for its derivative $f'(z)=-\frac{1}{z^2}$, if $z\neq 0$; $\therefore$ the function is analytic everywhere except at the point $z=0$, which is a singular point of $f(z)$.

If the derivative exists throughout some open two-dimensional region, the function is analytic in that region. Another method of determining whether a function is analytic or not is explained in the next article.

Note. If a function $f(z)$ is analytic, it is implied that it is single-valued.

2.51. Necessary and sufficient conditions for a function to be analytic.

Theorem. *The necessary and sufficient conditions that the continuous single-valued function* $f(z)=u(x, y)+iv(x,y)$ *is an analytic function of* $z=x+iy$ *in a given finite region* R *are that the four partial derivatives* $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ *exist, are continuous and satisfy the partial differential equations* $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ *at all points of* R.

Conditions are necessary. Let $w=f(z)$, then if w is analytic in R, it has a uniquely determined derivate w.r.t. z at any pt z in R,

i.e., $\underset{\triangle z \longrightarrow 0}{\text{Lt}} \left[\frac{\triangle w}{\triangle z}=\frac{f(z)+\triangle z)-f(z)}{\triangle z}\right]$ exists and is independent of the manner in which $\triangle z\to 0$. $\triangle z$ may, therefore, be conveniently chosen as a real or as a purely imaginary variable. The increment $\triangle z=\triangle x+i\triangle y$ becomes in the first case $\triangle x$ and in the second case $i\ \triangle y$. We may, therefore, write

$$\underset{\triangle z \longrightarrow 0}{\text{Lt}} \frac{\triangle w}{\triangle z}=\underset{\triangle x \longrightarrow 0}{\text{Lt}} \frac{\triangle w}{\triangle x}=\underset{\triangle y \longrightarrow 0}{\text{Lt}} \frac{\triangle w}{i\ \triangle y} \qquad \ldots\ldots(1)$$

Since the first limit exists, the second and third limits must also exist.

Thus, we have

$$\frac{dw}{dz}=\frac{\partial w}{\partial x}=\frac{1}{i}\frac{\partial w}{\partial y} \qquad \ldots\ldots(2)$$

However, we have

$$\frac{\partial w}{\partial x}=\frac{\partial}{\partial x}\ [u\,(x,y)+i\,v\,(x,y)]=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}, \qquad \ldots\ldots(3)$$

$$\text{and } \frac{\partial w}{\partial y}=\frac{\partial}{\partial y}\,[u\,(x,y)+i\,v\,(x,y)]=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}. \qquad \ldots\ldots(4)$$

Since $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ exist, it follows that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ all exist. Substituting the values of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ from (3) and (4) in (2), we get $\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$.

Equating the real and imaginary parts, we have

$$\left.\begin{aligned}&\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},\\ &i.\,e.,\quad u_x=v_y,\ u_y=-v_x.\end{aligned}\right\} \qquad \ldots\ldots(5)$$

These partial differential equations are known as *Cauchy-Riemann differential equations* (or *conditions*).

Conditions are sufficient. If we further assume that the four partial derivatives are also continuous*, we can show that the condition (5) are sufficient as well.

By the mean-value theorem, we have

$$\begin{aligned}\Delta u&=u\,(x+\Delta x,\ y+\Delta y)-u\,(x,\ y)\\ &=u\,(x+\Delta x,\ y+\Delta y)-u\,(x,y+\Delta y)+u\,(x,y+\Delta y)-u(x,y)\\ &=\Delta x.\ u_x\,(x+\theta_1\ \Delta x,\ y+\Delta y)+\Delta y.\ u_y\,(x,y+\theta_2\Delta x),\end{aligned}$$

where $0<\theta_1<1,\ 0<\theta_2<1$.

Since u_x, u_y are both continuous, we have

* This will be established in § 4.31.

$\triangle u = \triangle x.\ (u_x+\varepsilon_1)+\triangle y.\ (u_y+\varepsilon_2)$, where ε_1 and ε_2 both approach zero as $\triangle z \to 0$.

Similarly, $\triangle v = \triangle x.\ (v_x+\varepsilon_3)+\triangle y.\ (v_y+\varepsilon_4)$, where ε_3 and ε_4 both approach zero as $\triangle z \to 0$.

Hence, $\triangle w = \triangle u + i\ \triangle v$

$$= \left(\frac{\partial u}{\partial x}\triangle x+\frac{\partial u}{\partial y}\ \triangle y+\varepsilon_1\ \triangle x+\varepsilon_2\ \triangle y\right)$$

$$+i\left(\frac{\partial v}{\partial x}\triangle x+\frac{\partial v}{\partial y}\ \triangle y+\varepsilon_3\ \triangle x+\varepsilon_4\ \triangle y\right).$$

Assuming now that the Cauchy-Riemann conditions are satisfied at the point (x,y), we can replace $\frac{\partial u}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$, and then $\triangle w = \frac{\partial u}{\partial x}\ (\triangle x+i\ \triangle y)+i\ \frac{\partial v}{\partial x}\ (\triangle x+i\ \triangle y)$

$$+\delta_1\ \triangle x+\delta_2\ \triangle y,$$

where δ_1 and δ_2 both $\to 0$ as $\triangle z \to 0$

It follows that

$$\frac{\triangle w}{\triangle z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+\delta_1\ \frac{\triangle x}{\triangle z}+\delta_2\frac{\triangle y}{\triangle z}. \qquad \ldots\ldots(6)$$

Since $|\ \triangle x\ | \leqslant |\ \triangle z\ |$ and $|\ \triangle y\ | \leqslant |\ \triangle z\ |$;

$$\therefore \left|\ \frac{\triangle x}{\triangle z}\ \right| \leqslant 1, \text{ and } \left|\ \frac{\triangle y}{\triangle z}\ \right| \leqslant 1,$$

so that $\left|\ \delta_1\frac{\triangle x}{\triangle z}+\delta_2\ \frac{\triangle y}{\triangle z}\ \right| \leqslant |\ \delta_1\ | + |\ \delta_2\ | \to 0$, as $\triangle z \to 0$.

Hence, $\frac{dw}{dz}=\underset{\triangle z \longrightarrow 0}{\text{Lt}}\ \frac{\triangle w}{\triangle z}=\frac{\partial u}{\partial x}+\ i\ \frac{\partial v}{\partial x}$,

i.e. the derivative of w exists and the theorem is proved.

Note. The results of assuming differentiability are much more far-reaching than those of assuming continuity. Not only do the functions $u(x,\ y)$ and $v(x,\ y)$ possess partial derivatives of the first order but also they are connected by Cauchy-Riemann differential equations. Another distinguishing feature of differentiability is that it is not equivalent to the differentiability of $u(x,\ y)$ and $v(x,\ y)$ as is the case with continuity.

Ex. 1. *Show that the functions* $|z|^2$ *is not analytic at any point.*

Let $w=|z|^2$, *i. e.* $u+iv=x^2+y^2$, so that $u=x^2+y^2$, and $v=0$.

Now, $\frac{\partial u}{\partial x}=2x$, $\frac{\partial v}{\partial y}=0$, $\frac{\partial u}{\partial y}=2y$, $\frac{\partial v}{\partial x}=0$. Thus, the partial derivatives being polynomials in x and y are continuous everywhere. However, Cauchy-Riemann conditions are satisfied only at the origin but not in any neighbourhood of $z=0$. Hence, the function is not analytic at any point.

Note. $f'(z)$ exists only at $z=0$ as explained in § 2.4.

2.52. Polar form of Cauchy-Riemann conditions.

When $x=r\cos\theta$; $y=r\sin\theta$, we have

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}.$$

Similarly, $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$,

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}\cos\theta-\frac{\partial v}{\partial \theta}\frac{\sin\theta}{r},$$

and $$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}\sin\theta+\frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}.$$

Hence, the equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ when expressed in the polar form, become

$$\frac{\partial u}{\partial r}\cos 0-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}=\frac{\partial v}{\partial r}\sin\theta+\frac{\partial v}{\partial \theta}\frac{\cos 0}{r}, \qquad \ldots\ldots(1)$$

and $$\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}=-\frac{\partial v}{\partial r}\cos\theta+\frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}. \qquad \ldots\ldots(2)$$

Multiplying (1) by $\cos\theta$ *and* (2) by $\sin\theta$, and adding, we have

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}. \qquad \ldots\ldots(3)$$

Multiplying (1) by $-\sin\theta$ and (2) by $\cos\theta$, and adding, we have

$$\frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial v}{\partial r}. \qquad \ldots\ldots(4)$$

Equations (3) and (4) are the Cauchy-Riemann conditions in the polar form. The derivative in the polar form is then obtained as follows :

$$\frac{dw}{dz}=\frac{\partial w}{\partial x}=\frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial w}{\partial 0}\cdot\frac{\partial 0}{\partial x}$$

$$=\frac{\partial w}{\partial r}\cos 0-\left(\frac{\partial u}{\partial 0}+i\frac{\partial v}{\partial 0}\right)\frac{\sin 0}{r}$$

$$=\frac{\partial w}{\partial r}\cos 0-\left(-r\frac{\partial v}{\partial r}+ir\frac{\partial u}{\partial r}\right)\frac{\sin 0}{r}\text{ [From (3) and (4)]}$$

$$=\frac{\partial w}{\partial r}\cos 0-i\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)\sin 0$$

$$=(\cos 0-i\sin 0)\frac{\partial w}{\partial r}.$$

2.53. Function of a Function.

If $w=F(\zeta)$ and $\zeta=f(z)$ are analytic functions of ζ and z respectively, then w is an analytic function of z; for, if $\frac{dw}{d\zeta}$ and $\frac{d\zeta}{dz}$ exist, then $\frac{dw}{dz}$ being equal to their product also exists.

2.54. Inverse Functions. Let z_0 be a regular point of the analytic function $w=f(z)$ at which it takes the value w_0, then z can be regarded as a function of w with z_0 corresponding to w_0, *i.e.* $z=F(w)$. If $f'(z_0)\neq 0$, then w_0 is a regular point of $z=F(w)$ That this condition is necessary* is obvious. For, if $F(w)$ were analytic in a neighbourhood of w_0, $F'(w_0)$ would be finite; since, however, $F'(w_0)=\frac{1}{f'(z_0)}$, $f'(z_0)$ cannot be zero.

The function $z=F(w)$ is called the *inverse function* of $w=f(z)$.

***Note.** The proof of the sufficiency of the condition is beyond the scope of this book.

Ex. 1. *If* $f(z)=x^3y(y-ix)/(x^6+y^2)$, $z\neq 0$; $f(0)=0$, *prove that* $[f(z)-f(0)]/z\to 0$ *as* $z\to 0$ *along any radius vector but not as* $z\to 0$ *in any manner.* *(Agra, 1945)*

Let z move along $y=mx$, then

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\frac{-ix^3y(x+iy)}{x^6+y^2}\cdot\frac{1}{z}$$

$$=\lim_{x\to 0}\frac{-imx^4}{x^6+m^2x^2}\quad(\text{since } y=mx.)$$

$=0.$

If z moves along the curve $y=x^3$,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{x\to 0}\frac{-ix^6}{2x^6}=-\frac{i}{2}.$$ Hence, the result.

Important Notes. $f(z)$ may not be analytic if we merely know that $\frac{f(z)-f(z_0)}{z-z_0}\to$ a limit along two straight lines at right angles. Actually the definition fails if we restrict ourselves to any special class of paths.

Ex. 2. *Prove that the fuction* $u+iv=f(z)$, *where*

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}\quad(z\neq 0),\ f(0)=0$$

is continuous and that the Cauchy-Riemann conditions are satisfied at the origin, yet $f'(0)$ *does not exist.* (*Agra 1946, 56*)

Here $u=\frac{x^3-y^3}{x^2+y^2}$, and $v=\frac{x^3+y^3}{x^2+y^2}$. At the origin, $u=0$ and $v=0$, since $f(0)=0$. Hence, $f(z)$ is continuous everywhere as both u and v are continuous everywhere.

At the origin, $\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{x}{x}=1,$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\frac{-y}{y}=-1,$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\frac{x}{x}=1,$$

and $$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y}=\lim_{y\to 0}\frac{y}{y}=1.$$

Hence, the Cauchy-Riemann conditions are satisfied at $z=0$.

Again, $f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\left[\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}\cdot\frac{1}{x+iy}\right]$

$$= \underset{z\to 0}{\text{Lt}} \frac{(1+i)\,(x^3+iy^3)\,(x-iy)}{(x^2+y^2)^2}$$

$$= \underset{z\to 0}{\text{Lt}} \frac{(1+i)\,[x^4+y^4+ixy(y^2-x^2)\,]}{(x^2+y^2)^2}.$$

Now, let z move along $y=x$, then $f'(0)=\frac{1}{2}\,(1+i)$.

Also, $f'(0)=\left(\dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}\right)_{(0,\,0)}=1+i.$

Hence, $f'(0)$ is not unique and, therefore, does not exist.

Note. The partial derivatives of the first order u and v must be continuous, otherwise the function will not be analytic. In this example, the partial derivatives of the first order of u and v are not continuous. Thus, the function $f(z)$ is not analytic at the origin.

Ex. 3. *Show that the function* $f(z)=e^{-z^{-4}}$ $(z\neq 0)$, $f(0)=0$ *is not analytic at* $z=0$, *although the Cauchy-Riemann conditions are satisfied at that point.* (*Agra 1936, 54*)

Here $f'(z)=4z^{-5}\,e^{-z^{-4}}$, which exists for all finite values of z except $z=0$.

At $z=0$, $\dfrac{\partial u}{\partial x}=\underset{x\to 0}{\text{Lt}}\dfrac{u(x,\,0)-u(0,\,0)}{x}=\underset{x\to 0}{\text{Lt}}\dfrac{e^{-x^{-4}}}{x}=0,$

$$\frac{\partial u}{\partial y}=\underset{y\to 0}{\text{Lt}}\frac{e^{-y^{-4}}}{y}=0,\quad \frac{\partial v}{\partial x}=\underset{x\to 0}{\text{Lt}}\frac{0}{x}=0,$$

and $\dfrac{\partial v}{\partial y}=\underset{y\to 0}{\text{Lt}}\dfrac{0}{y}=$; so that the Cauchy-Riemann conditions are satisfied.

Inspite of this, $f(z)$ is not analytic at $z=0$. For, suppose that $z=re^{i\pi/4}$, then $f(z)=e^{-r^{-4}}\,e^{-i\pi}=e^{r^{-4}}$, which tends to infinity as $r\to 0$. Thus, $f(z)$ is not continuous at $z=0$, since $f(0)=0$ and hence cannot be analytic there.

Ex. 4. *For what values of* z *do the functions defined by the following equations cease to be analytic* ?

(i) $z=\log \rho+ i\,\varnothing$, where $w=\rho\,(\cos\varnothing+i\sin\varnothing)$.

(ii) $z=\sinh u\cos v+ i\cosh u\sin v$, where $w=u+iv$.

(i) $\dfrac{dz}{dw}=(\cos\varnothing-i\sin\varnothing)\dfrac{\partial z}{\partial\rho}$ [See § 2. 52]

$$=\frac{1}{\cos\varnothing+i\sin\varnothing}\cdot\frac{1}{\rho}\Big[\text{since } z=\log\rho+i\varnothing,\ \text{giving } \frac{\partial z}{\partial\rho}=\frac{1}{\rho}.\Big]$$

$$=\frac{1}{w}.$$

Hence, we conclude that w is an analytic function of z so long as $\frac{dz}{dw}\neq 0$, *i.e.* so long as w is not infinite [§ 2.54].

Now w cannot be infinite unless ρ is infinite, *i.e.* z is infinite. Thus, there is no point in any finite region where $\frac{dz}{dw}=0$. Hence, w is an analytic function in any finite region.

(ii) $z=\sinh u\cos v+i\cosh u\sin v$.

Here $\frac{dz}{dw}=\frac{\partial z}{\partial u}=\cosh u\cos v+i\sinh u\sin v$.(1)

Now $z^2=(\cosh^2u-1)\cos^2v-(1+\sinh^2u)\sin^2v$
$+2i\sinh u\cosh u\sin v\cos v$
$=(\cosh u\cos v+i\sinh u\sin v)^2-(\cos^2v+\sin^2v)$
$=\left(\frac{dz}{dz}\right)^2-1$, *i.e.*, $\frac{dz}{dw}=\pm\sqrt{z^2+1}$

Hence, w ceases to be analytic when $\frac{dz}{dw}=0$, *i. e.*, when $z=\pm i$.

2.6. Harmonic Functions. Let the function $f(z)=u+iv$ be analytic in some region of the z-plane, then

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad(1)$$

and, therefore,

$$\frac{\partial^2u}{\partial x^2}=\frac{\partial^2v}{\partial x\partial y},\ \frac{\partial^2u}{\partial y^2}=-\frac{\partial^2v}{\partial y\partial x}. \qquad(2)$$

Assuming that partial derivatives of u and v of all orders exist and are continuous functions of x and y,* it follows that

* This will be established in § 4.31.

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x};$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \ldots\ldots(3)$$

throughout the region. Equation (3) is known as *Laplace's Differential Equation* and is of prime importance in theoretical physics. Any function that has continuous partial derivatives of the first and second order and that satisfies Laplace's equation is called a *Harmonic Function.*

Again, differentiating the first equation in (1) w.r.t. y, the second w.r.t. x and subtracting, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \qquad \ldots\ldots(4)$$

which shows that v is also a Harmonic function.

Hence, if $f(z) = u + iv$ be analytic both u and v satisfy Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

and are called *conjugate harmonic functions* (or simply *conjugate functions*). It may be noted here that this is a different use of the word conjugate from that employed in defining $\bar{z}$.

It also follows from equations (1) that

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0 \qquad \ldots\ldots(5)$$

The geometrical interpretation of this relation is that the families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ in the xy—plane, intersect at right angles at all their points of intersection. For, from $u(x, y) = c_1$, we have $\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$, or $\frac{dy}{dx} = -\frac{\partial u}{\partial x} / \frac{\partial u}{\partial y}$.

Similarly, from $v(x, y) = c_2$ we have $\frac{dy}{dx} = -\frac{\partial v}{\partial x} / \frac{\partial v}{\partial y}$. If the

curves $u=c_1$ and $v=c_2$ intersect at right angles, then $\left(-\frac{\partial u}{\partial x}/\frac{\partial u}{\partial y}\right)$ $\left(-\frac{\partial v}{\partial x}/\frac{\partial v}{\partial y}\right)=-1$, which reduces to (5).

Thus, the families of curves $u=c_1$, $v=c_2$ are *orthogonal* families.

2.61. Determination of the conjugate function. If one of the two conjugate functions be given, the other can be determined by the use of Cauchy-Riemann conditions.

For example, if $u(x, y)$ is a harmonic function in a given finite region, then

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$$=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy \text{ is a perfect differential,}$$

since $\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$, from §2·6(3);

and the conjugate function v can, therefore, be determined.

2.62. Determination of the analytic function $f(z)=u+i\,v$, when either u or v is given.

Method I (*due to Milne Thompson*). Since $x=\frac{1}{2}(z+\bar{z})$, $y=\frac{1}{2i}(z-\bar{z})$, $f(z)$ can be regarded as a formal identity in two independent variables z, $\bar{z}$. Putting $z=\bar{z}$, we have $x=z$, and $y=0$;

$\therefore\quad f(z)\equiv u(z, 0)+i\,v(z, 0)$.

Now, $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$, by Cauchy-Riemann conditions.

Hence, if we write $\frac{\partial u}{\partial x}=\varnothing_1(x,y)$, and $\frac{\partial u}{\partial y}=\varnothing_2(x,y)$, we have

$f(z)=\int\{\varnothing_1(z,0)-i\,\varnothing_2(z, 0)\}\,dz+c$, where c is an arbitrary constant.

Similarly, if $v(x,y)$ is given, it can be shown that

$$f(z) = \int \{\psi_1(z, 0)+i\,\psi_2(z,0)\}\,dz+c,$$

where $\psi_1(x,y)=\frac{\partial v}{\partial y}$, $\psi_2(x,y)=\frac{\partial v}{\partial x}$ and c is an arbitrary constant.

Method II. (*due to A. Oppenheim*). If only the constants occuring in $f(z)$ be replaced by their conjugates, let the function obtained be denoted by $\bar{f}(z)$. It follows that $\bar{f}(\bar{z})=\overline{f(z)}$, where $\bar{z}$ is the conjugate of z and $\overline{f(z)}$ the conjugate of $f(z)$ as usual.

Thus, if $f(z)=u(x,y)+iv(x,y)$, then $\bar{f}(\bar{z})=u(x,y)-iv(x,y)$

Hence, $2u=f(z)+f(\bar{z})$, $2iv=f(z)-f(\bar{z})$.

Now, regarding $f(z)$ as a formal identity in z and $\bar{z}$ as in Method I above,

we have $$f(z)+f(\bar{z})=2\,u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right), \qquad \ldots\ldots(1)$$

and $$f(z)-f(\bar{z})=2\,iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right). \qquad \ldots\ldots(2)$$

Thus, when either u or v is known as a real function of x and y, the function $f(z)$ can be determined by inserting in (1) or (2) a convenient value of $\bar{z}$.

Putting $\bar{z}=0$, we have $f(z)=2u\left(\frac{z}{2}, \frac{z}{2i}\right)-c_1$, $\qquad \ldots\ldots(3)$

and $f(z)=2iv\left(\frac{z}{2}, \frac{z}{2i}\right)+c_1$, $\qquad \ldots\ldots(4)$

where c_1 is a constant.

Hence, if $u\left(\frac{z}{2}, \frac{z}{2i}\right)$, and $v\left(\frac{z}{2}, \frac{z}{2i}\right)$ are determinate, the relation (3) or (4) can be used to find out $f(z)$, but see Ex. 4 below.

Ex. 1. *Given $u=x^3-3xy^2$, show that there exists a function $v(x,y)$ such that $w=u+iv$ is analytic in a finite region. Determine the function $v(x,y)$.* (*Agra 1949, '51*)

The given function satisfies Laplace's equation ; for, we have

$$\frac{\partial u}{\partial x}=3x^2-3y^2,\ \frac{\partial u}{\partial y}=-6xy,\ \frac{\partial^2 u}{\partial x^2}=6x,\ \frac{\partial^2 u}{\partial y^2}=-6x\ ;$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

By using Cauchy-Riemann conditions, we see that

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$$

$$=6xy\ dx+(3x^2-3y^2)\ dy.$$

Solving this perfect differential equation, we have

$v=3x^2y-y^3+c$, where c is an arbitrary constant.

Hence, the harmonic conjugate of the function $u=x^3-4xy^2$ is $v=3x^2y-y^3+c$. The corresponding function $f(z)=u+iv$ is

$$f(z)=x^3-3xy^2+i(3x^2y-y^3+c) \qquad \ldots\ldots(1)$$

$$=(x+iy)^3+ic=z^3+ic. \qquad \ldots\ldots(2)$$

This form is suggested by noting that when $y=0$, equation (1) becomes $f(x)=x^3+ic$.

From the form (2), it will be at once seen that it is analytic in any finite region.

Note. The analytic fuction $f(z)=u+iv$ corresponding to the harmonic function can be easily determined by §2.62 Method II.

We have $f(z)=2u\left(\frac{z}{2},\frac{z}{2i}\right)+c_1$, where c_1 is a constant

$$=2\left[\left(\frac{z}{2}\right)^3-3\frac{z}{2}\left(\frac{z}{2i}\right)^2\right]+c_1$$

$$=2\left(\frac{z^3}{8}+\frac{3z^3}{8}\right)+c_1=z^3+c_1.$$

Ex. 2. *Show that the function* $u=e^x(x\ cos\ y-y\ sin\ y)$ *satisfies Laplace's equation and find the corresponding analytic function* $u+iv$.

(Agra 1957 ; Vikram 1959)

Method I. Let $\frac{\partial u}{\partial x}=e^x(x\cos y-y\sin y)+e^x\cos y=\phi_1(x,y)$,

and $\frac{\partial u}{\partial y}=e^x(-x\sin y-y\cos y-\sin y)=\phi_2(x,y)$

then the corresponding analytic function $f(z)=u+iv$ is given by

$f(z)=\int\{\phi(z,0)-i\ \phi_2\ (z,0)\}\ dy+c$, where c is an arbitrary constant.

$$=\int e^z(z+1)\ dz+c=ze^z+c.$$

Method II. $f(z)= 2u\left(\frac{z}{2}, \frac{z}{2i}\right)+c_1$, where c_1 is a constant

$$= 2\left[e^{z/2}\left(\frac{z}{2}\cos\frac{z}{2i}-\frac{z}{2i}\sin\frac{z}{2i}\right)\right]+c_1$$

$$=z\,e^{z/2}\left(\cos\frac{z}{2i}+i\sin\frac{z}{2i}\right)+c_1=ze^{z/2}.\ e^{iz/2i}+c_1$$

$$=ze^{z}+c_1.$$

Ex. 3. *Prove that, if* $u=x^2-y^2$, $v=-y/(x^2+y^2)$, *both* u *and* v *satisfy Laplace's equation, but that* $u+iv$ *is not an analytic function of* z.

We have

$$\frac{\partial u}{\partial x}=2x,\ \frac{\partial^2 u}{\partial x^2}=2,\ \frac{\partial u}{\partial y}=-2y,\ \frac{\partial^2 u}{\partial y^2}=-2\ ;$$

$$\frac{\partial v}{\partial x}=\frac{2xy}{(x^2+y^2)^2},\ \frac{\partial^2 v}{\partial x^2}=\frac{2y(y^2-3x^2)}{(x^2+y^2)^3},\ \frac{\partial v}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial^2 v}{\partial y^2}=\frac{2y\,(3x^2-y^2)}{(x^2+y^2)^3}\ ;$$

$$\therefore\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ \text{and}\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

Hence, both u and v satisfy Laplace's equation. Still $u+iv$ is not an analytic function of z, as Cauchy-Riemann conditions are not satisfied, since $\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}\neq-\frac{\partial v}{\partial x}$.

Ex. 4. *If* $f(z)=u+iv$ *is an analytic function of* $z=x+iy$, *and* $u-v=\dfrac{\cos x+\sin x-e^{-y}}{2\cos x-e^{y}-e^{-y}}$; *find* $f(z)$ *subject to the condition* $f(\pi/2)=0$.

(*Agra 1943*)

Method I. If $u+iv=f(z)$, $iu-v=if(z)$, and by addition,

$u-v+i(u+v)=(1+i)f(z)=F(z)$, say(1)

Hence, $U=u-v$ and $V=u+v$ are conjugate functions, being the real and imaginary parts of the analytic function $F(z)$. Simplifying the given expression for $u-v$, we have

$$U=\tfrac{1}{2}\left[1+\frac{\sin x+\sinh y}{\cos x-\cosh y}\right],\qquad \text{.....(2)}$$

from which we get $\frac{\partial U}{\partial y}=\frac{1}{2}\cdot\frac{1+\sin x\sinh y+\cos x\cosh y}{(\cos x-\cosh y)^2}$

$$=\phi_1(x,y),$$

and $\frac{\partial U}{\partial y}=\frac{1}{2}\cdot\frac{-1+\sin x\sinh y+\cos x\cosh y}{(\cos x-\cosh y)^2}=\phi_2(x,y)$;

$\therefore\quad F(z)=\int\{\phi_1(z,0)-i\phi_2(z,0)\}\,dz+c$, where c is an arbitrary constant.

$$=\int\left(\frac{1}{2}\cdot\frac{1}{1-\cos z}+\frac{i}{2}\cdot\frac{1}{1-\cos z}\right)dz+c$$

$$=\tfrac{1}{4}(1+i)\int\operatorname{cosec}^2\frac{z}{2}\,dz+c$$

$$=-\tfrac{1}{2}(1+i)\cot\frac{z}{2}+c,$$

and $\therefore$ from (1), $f(z)=-\frac{1}{2}\cot\frac{z}{2}+c'$, where c' is a constant such that $f(\pi/2)=0=-\frac{1}{2}+c'$, *i.e.*, $c'=\frac{1}{2}$.

Hence, $f(z)=\frac{1}{2}\left(1-\cot\frac{z}{2}\right)$.

Method II. We write F (z), U and V as above. In (2), if we put $x=\frac{z}{2}$, and $y=\frac{z}{2i}$ the denominator of the fraction becomes $\cos\frac{z}{2}-\cosh\frac{z}{2i}=0$; so we cannot use § 2.62 (3). From §2.62 (1),

$$F(z)=2U\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)-\bar{F}(\bar{z}).$$

A convenient value for $\bar{z}$ in this identity is π, and this gives

$$F(z)=1-\bar{F}(\pi)+\frac{\sin\frac{1}{2}(z+\pi)+\sinh\frac{1}{2i}(z-\pi)}{\cos\frac{1}{2}(z+\pi)-\cosh\frac{1}{2i}(z-\pi)}$$

$$=c_1+\left(\cos\frac{z}{2}+\frac{1}{i}\sin\frac{z-\pi}{2}\right)\Big/\left(-\sin\frac{z}{2}-\cos\frac{z-\pi}{2}\right)$$

$$=c_1+\frac{(1+i)\cos\frac{z}{2}}{-2\sin\frac{z}{2}}=c_1-\tfrac{1}{2}(1+i)\cot\frac{z}{2}.$$

If follows that $f(z)=k-\frac{1}{2}\cot\frac{z}{2}$, where the constant k is found by putting $z=\frac{\pi}{2}$, which gives $0=k-\frac{1}{2}$.

Hence, $f(z)=\frac{1}{2}\left(1-\cot\frac{z}{2}\right)$.

Exercise 2 (a)

1. Find the loci of points z satisfying the following relations :

(i) $0<\text{Re }(iz)<2\pi$. (ii) $\text{Re }(z^2)=a\ (\gtreqless 0)$.

(iii) $\text{Im }(z^2)=a\ (\lessgtr 0)$. (iv) $|z^2-z|\leqslant 1$.

(v) $|z^2-1|=a>0$. (vi) $\left|\frac{1}{z}\right|<d,\ d>0$.

(vii) $\left|\frac{z-1}{z+1}\right|\leqslant 1$. (viii) $\left|\frac{z-z_1}{z-z_2}\right|=1$.

(ix) $|z-a|+|z-b|\leqslant k,\ k>0$.

2. If $(z)=\frac{xy^2(x+iy)}{x^2+y^4}\ (z\neq 0), f(0)=0$; prove that $\frac{f(z)-f(0)}{2}\to 0$ as $z\to 0$ along any radius vector, but not as $z\to 0$ in any manner.

3. Show that the function $f(z)=\sqrt{|xy|}$ is not analytic at the origin, although the Cauchy-Riemann conditions are satisfied at that point.

4. If n is real, show that $r^n(\cos n\theta+i\sin n\theta)$ is analytic except possibly when $r=0$, and that its derivative is $n\,r^{n-1}[\cos(n-1)\theta+i\sin(n-1)\theta]$.

5. For what values of z do the functions w defined by the following equations cease to be analytic ?
(i) $z=e^{-v}(\cos u+i\sin u)$. (ii) $z=\sin u\cosh v+i\cos u\sinh v$.

6. If $f(z)=u+iv$ is an analytic function of $z=x+iy$ and ψ is any function of x and y with differential coefficients of the first two orders, show that

(i) $\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2=\left\{\left(\frac{\partial\psi}{\partial u}\right)^2+\left(\frac{\partial\psi}{\partial v}\right)^2\right\}|f'(z)|^2$;

and (ii) $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\left(\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right)|f'(z)|^2$.

(Agra 1937, '54)

7. If $f(z)$ is an analytic function of $z=x+iy$, prove that

(i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$;

(ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p=p^2|f(z)|^{p-2}|f'(z)|^2$;

and (iii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|n|^p=p(p-1)|u|^{p-2}|f'(z)|^2$.

8. If $w=f(z)$ is an analytic function of z such that $f'(z)\neq 0$, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=0.$$

If $|f'(z)|$ is the product of a function of x and a function of y, show that

$$f'(z)=e^{az^2+\beta z+\gamma},$$

where a is real and β and γ are complex constants.

9. Prove that $u=y^3-3x^2y$ is a harmonic function. Determine its harmonic conjugate and find the corresponding analytic function in terms of z.

10. Prove that the following functions satisfy Laplace's equation, and determine the corresponding analytic function $u+iv$ in each case :

(i) $u=x^3-3xy^2+3x^2-3y^2+1$. *(Agra 1955)*

(ii) $u=\sin x\cosh y+2\cos x\sinh y+x^2-y^2+4xy$.

11. If ϕ and ψ are functions of x and y satisfying Laplace's equation, show that $s+it$ is analytic, where $s=\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x}$ and $t=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}$. *(Agra 1936)*

12. If $f(z)=u+iv$ is an analytic function of $z=x+iy$, and $u-v=e^x(\cos y-\sin y)$, find $f(z)$ in terms of z. (*Agra 1952*)

13. If $(z)=u+iv$ be an analytic function of $z=x+i\,y$, and $u-v=(x-y)(x^2+4xy+y^2)$, find $f(z)$ in terms of z. (*Agra 1942*)

14. Determine the analytic function $f(z)=u+iv$, given that $u+v=(2\sin 2x)/(e^{2y}+e^{-2y}-2\cos 2x)$.

15. If $f(z)=u+iv$ is an analytic function of $z=x+iy$, and $u-v=(e^y-\cos x+\sin x)/(\cosh y-\cos x)$

find $f(z)$ subject to the condition $f(\pi/2)=\dfrac{3-i}{2}$.

16. Using Euler's theorem that, if u is a function of x, y which is homogeneous of degree m, $x\,u_x+y\,u_y=m\,u$, show that, if u also satisfies Laplace's equation, it is the real part of a function $f(z)$ such that $f(z)=x+\dfrac{i}{m}(yu_x-xu_y)$.

Determine $f(z)$, when

(i) $u=x^3-3\,x\,y^2$;

(ii) $v=\dfrac{y}{x^2+y^2}$;

(iii) $u=ax^2+2bxy-ay^2$;

(iv) $u=\dfrac{xy}{x^4+y^4}$.

17. If $f(z)$ is analytic in a domain and $f'(z)\neq 0$, $|f(z)|\neq 1$, show that the function $w=\log\dfrac{|f'(z)|}{1-|f'(z)|^2}$ satisfies the partial differential equation $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=4e^{2w}$.

2.7. Power Series. An infinite series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, where the variable z and the constants z_0, a_0, a_1, a_2,... are, in general, complex numbers, is called a *power series*. If $z_0=0$, the power series reduces to the simple form $\sum_{n=0}^{\infty} a_n z^n$, ...(1) which may be considered as a typical power series.

The convergence of the series (1) can be discussed exactly in the same way as in the case of a series of real terms. The absolute

convergence of the series (1) follows immediately from the convergence of $\sum_0^\infty |a_n| \, |z|^n$, which is a series of positive real terms.

Let $\underset{n \longrightarrow \infty}{\text{Lt}} |a_n|^{\frac{1}{n}} = \frac{1}{R}$, then by Cauchy's root test, the series (1) is absolutely convergent if $|z| < R$ and divergent if $|z| > R$. The number R is called the *radius of convergence* and the circle $|z| = R$ is called the *circle of convergence* of the power series. The different cases to be considered are :

(i) $R=0$, (ii) R finite, and (iii) R infinite.

When $R=0$, the series is convergent only when $z=0$ and the case is trivial. When R is infinite, the series is convergent for all finite valves of z. When R is finite, the series is convergent at all points within the circle $|z| = R$ and divergent at all points outside $|z| = R$.

We shall now show that within its circle of convergence, a power series represents an analytic function. Later on (§4.5), we shall see that, conversely, an analytic function can be represented by means of a power series. This shows the great importance of power series in the evaluation of analytic functions.

Theorem. *If a power series has a finite non-zero radius of convergence, its sum is an analytic function at every point within its circle of convergence.*

Let $f(z) = \sum_0^\infty a_n z^n$, whose radius of convergance R, is a non-zero finite number, and let $g(z) = \sum_1^\infty n a_n z^{n-1}$.(2)

Now, if $0 < \rho < R$, $a_n \rho^n$ is bounded, *i.e.* $|a_n \rho^n| < M$, say. Further, let $|z| = r$, $|h| = \varepsilon$, then if $r < \rho$ and $r + \varepsilon < \rho$,

$$\frac{f(z+h)-f(z)}{h} - g(z) = \sum_0^\infty \left[a_n \left\{ \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right\} \right].$$

Now $\left| \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right| = \left| \frac{n(n-1)}{1.2} z^{n-2} h + \ldots + h^{n-1} \right|$

$$\leqslant \frac{n(n-1)}{1.2} r^{n-2}\varepsilon + \ldots + \varepsilon^{n-1} = \frac{(r+\varepsilon)^n - r^n}{\varepsilon} - nr^{n-1}.$$

Hence, $$\left| \frac{f(z+h)-f(z)}{h} - g(z) \right| = \left| \sum_0^\infty a_n \left[\frac{(z+h)^n - z^n}{h} - nz^{n-1} \right] \right|$$

$$\leqslant \sum_0^\infty |a_n| \left[\frac{(r+\varepsilon)^n - r^n}{\varepsilon} - nr^{n-1} \right]$$

$$= \sum_0^\infty |a_n| \rho^n \left\{ \frac{\left(\frac{r+\varepsilon}{\rho}\right)^n - \left(\frac{r}{\rho}\right)^n}{\varepsilon} - \frac{n}{r}\left(\frac{r}{\rho}\right)^n \right\}$$

$$\leqslant M \sum_0^\infty \left[\frac{1}{\varepsilon} \left\{ \left(\frac{r+\varepsilon}{\rho}\right)^n - \left(\frac{r}{\rho}\right)^n \right\} - \frac{n}{r}\left(\frac{r}{\rho}\right)^n \right]$$

$$= M \left[\frac{1}{\varepsilon}\left(\frac{\rho}{\rho - r - \varepsilon} - \frac{\rho}{\rho - r}\right) - \frac{\rho}{(\rho - r)^2} \right] = \frac{M\rho\varepsilon}{(\rho - r - \varepsilon)(\rho - r)^2},$$

which tends to zero if $\varepsilon = |h| \to 0$. It follows that $f(z)$ has the derivative $g(z)$. This show that $f(z)$, which is obviously one-valued when $|z| < R$, is also differentiable. Hence $f(z)$ is analytic within $|z| = R$ and its derivative can be found by term by term differentiation.

Now, since $\underset{n \to \infty}{Lt} |na_n|^{\frac{1}{n}} = \underset{n \to \infty}{Lt} |a_n|^{\frac{1}{n}} = \frac{1}{R}, \left[\because Lt_n^{\frac{1}{n}} = 1 \right]$

it follows that the series (2) has the same radius of convergence R as the series (1) whose sum is $f(z)$. By applying the theorem to the derived series (2), we see that $g(z) = f'(z)$ is itself an analytic function in $|z| < R$ and that its derivative is $\sum_2^\infty n(n-1)a_n z^{n-2}$, and so on.

By continuing in this way, we can show that *a power series can be differentiated term by term as often as we please at any point within its circle of convergence.*

Note. A more general theorem is proved in § 4.42 by using the properties of uniformly convergent series.

2.8. Some Elementary Functions. We shall now consider briefly some elementary functions, which are easily seen to be analytic.

I. *Rational Functions.* Every polynomial in z

$$P(z)\equiv a_0+a_1z+a_2z^2+\ldots+a_nz^n,$$

is an analytic function for all values of z, since $P(z)$ can be regarded as a powers series which is convergent for all values of z.

The points at which a polynomial vanishes are called its zeros. Now assuming that a polynomial can have only a finite number of zeros, we see that the quotient of two polynomials (*i.e.* a *rational function*) $\frac{P(z)}{Q(z)}$ is analytic at all points except the zeros of $Q(z)$.

Note. The words 'analytic at all points or analytic in the whole z-plane' mean analytic for all finite values of z.

II. *The Exponential Function.* We define the exponential function of a complex variable in the same way as the exponential function of a real variable. The exponential function of $z=x+iy$ is denoted by $exp\ z$, or $E(z)$ and is defined by the power series

$$E(z)=1+\frac{z}{\lfloor 1}+\frac{z^2}{\lfloor 2}+\ldots=\sum_0^\infty \frac{z^n}{\lfloor n}. \qquad \ldots\ldots(1)$$

Since, by Ratio test, this series converges for all values of z, it follows that the series (1) defines a function analytic in the whole z-plane. Differentiating the series (2) term by term, we see that the derivative of $E(z)$ is $E(z)$, *i.e.* $E'(z)=E(z)$. Hence, if a be any finite constant, the derivative of the analytic function $f(z)=E(z).\ E(a-z)$ is zero. Thus, $f(z)$ is necessarily a constant, whose value is found to be $E(a)$ by putting $z=0$; $\therefore\ E(z).\ E(a-z)=E(a)$, or, putting $a-z=z_1$, $E(z)\,E(z_1)=E(z+z_1)$. $\ldots\ldots(2)$*

Because of this result, which is called the *addition theorem* of the exponential function, $E(z)$ is conveniently written as e^z, *i.e.*, $e^z=\sum_0^\infty \frac{z^n}{\lfloor n}$.

An important result which follows from the addition theorem (2) is that e^z never vanishes at any finite point of the complex plane For, we have $e^z.\ e^{-z}=e^0=1$;

*The addition theorem can also be proved by multiplying two absolutely convergent series.

since e^{-z} is a definite number for any finite value of z, it follows that e^z can never be zero.

It should be noted that e^z is a purely symbolic notation for the series (1) and it should never be regarded as the 'z th power' of e.

Further, if y is real,

$$e^{iy}=\sum_0^\infty \frac{(i\,y)^n}{\lfloor n}=\sum_0^\infty (-1)^n\,\frac{y^{2n}}{\lfloor 2n}+i\sum_0^\infty (-1)^n\,\frac{y^{2n+1}}{\lfloor 2n+1}$$

$$=\cos\, y+i\,\sin\, y=\text{cis}\, y,$$

since, for real values of y, $\cos y=1-\frac{y^2}{\lfloor 2}+\frac{y^4}{\lfloor 4}-\cdots,$

and $\sin y=y-\frac{y^3}{\lfloor 3}+\frac{y^5}{\lfloor 5}-\cdots.$

Hence, $e^z=e^{x+iy}=e^x.e^{iy}=e^x\,\text{cis}\, y$.

Thus, we see that the function e^z is a period function with the period $2\pi i$, since sin y and cos y have the period 2π, *i. e.* if n is any integer (positive or negative) or zero, then $e^z=e^{z+2\pi i}=e^{z+2n\pi i}$.

III. *The Trigonometrical Functions.* We define the trigometrical functions of a complex variable by means of a power series just as is the case of a real variable. Thus, cos z and sin z are defined by the power series

$$\cos\, z=1-\frac{z^2}{\lfloor 2}+\frac{z^4}{\lfloor 4}-\cdots\quad=\sum_0^\infty (-1)^n\,\frac{z^{2n}}{\lfloor 2n}, \qquad \ldots\ldots(3)$$

$$\text{and } \sin z=z-\frac{z^3}{\lfloor 3}+\frac{z^5}{\lfloor 5}-\cdots\quad=\sum_0^\infty (-1)^n\,\frac{z^{2n+1}}{\lfloor 2n+1}. \qquad \ldots\ldots(4)$$

It should be noted that cos z and sin z are symbolic notations for the series (3) and (4) respectively and they lose the meaning given to them for real values of z.

Since both the series (3) and (4) converge for all finite values of z, it follows that cos z and sin z are analytic in the whole z—plane.

Differentiating the series (3) and (4) term by term, we see that

$$\frac{d}{dz}(\cos z)=-\sin z, \text{ and } \frac{d}{dz}(\sin z)=\cos z.$$

By changing z to $-z$ in (3) and (4) we immediately see that

$$\cos(-z)=\cos z, \quad \text{and } \sin(-z)=-\sin z, \qquad \ldots\ldots(5)$$

i. e. cos z is an *even function* of z and sin z is an *odd function* of z.

Now, according to the definition of an exponential function, we have

$$e^{iz}=\sum_0^\infty \frac{(iz)^n}{\lfloor n} =\left(1-\frac{z^2}{\lfloor 2}+\frac{z^4}{\lfloor 4}\ldots\ldots\right)+i\left(z-\frac{z^3}{\lfloor 3}+\frac{z^5}{\lfloor 5}\ldots\ldots\right)$$

$$=\cos z+i\sin z. \qquad \ldots\ldots(6)$$

Changing z to $-z$ and applying (5), we get

$$e^{-iz}=\cos z-i\sin z. \qquad \ldots\ldots(7)$$

Multiplying (6) and (7) and using addition theorem (2), we have

$$\cos^2 z+\sin^2 z=1,$$

which shows that the fundamental identity of trigonometry is true for complex values of the variable.

From (6) and (7), we also obtain Euler's formulae

$$\cos z=\tfrac{1}{2}(e^{iz}+e^{-iz}), \quad \sin z=\frac{1}{2i}(e^{iz}-e^{-iz}).$$

These formulae can be used to give alternative definitions of cos z and sin z.

Other trigonometrieal functions are defined by the equations

$$\tan z=\frac{\sin z}{\cos z}, \quad \cot z=\frac{\cos z}{\sin z}, \quad \sec z=\frac{1}{\cos z}, \quad \operatorname{cosec} z=\frac{1}{\sin z}.$$

Clearly tan z, cot z, cosec z are odd functions of z and sec z is an even function of z.

From Euler's formulae and addition theorem (2) above, we get

$$\left.\begin{aligned}&\cos(z_1+z_2)=\cos z_1\cos z_2-\sin z_1\sin z_2,\\ \text{and}\quad &\sin(z_1+z_2)=\sin z_1\cos z_2+\cos z_1\sin z_2,\end{aligned}\right\} \qquad \ldots\ldots(8)$$

which shows that the addition theorems of trigonometry are true for complex values of the variables. It also follows that all the elementary identities of trigonometry, being algebraic deductions from the fundamental identity and the addition theorems, are true for complex values of the variables.

IV. The Hyperbolic Functions. These are defined in the same way as for real variables. Thus, cosh z, sinh z etc. are defined by the equations $\cosh z=\frac{1}{2}(e^z+e^{-z})$, $\sinh z=\frac{1}{2}(e^z-e^{-z})$,

$$\tanh z=\frac{\sinh z}{\cosh z},\ \operatorname{sech} z=\frac{1}{\cosh z},\ \operatorname{cosech} z=\frac{1}{\sinh z},$$

$$\coth z=\frac{1}{\tanh z}.$$

Since the power series for cosh z, *i. e.*, $\sum_0^\infty \frac{z^{2n}}{\lfloor 2n}$ is convergent for all values of z, it follows that cosh z is an analytic function in the whole z-plane.

Similarly, sin z is an analytic function in the whole z-plane.

The following important relations can be easily proved :

$$\cosh^2 z-\sinh^2 z=1,\ \frac{d}{dz}(\cosh z)=\sinh z,\ \frac{d}{dz}(\sinh z)=\cosh z,$$

$$\left.\begin{array}{ll}\sin iz=i\sinh z, & \cos iz=\cosh z,\\ \sinh iz=i\sin z, & \cosh iz=\cos z.\end{array}\right\} \quad \ldots\ldots(9)$$

With the help of the relations (9), the properties of the hyperbolic functions can be deduced from the corresponding properties of the trigonometrial functions.

Note 1. Functions which are analytic in every finite region of the $z-$plane are called *integral* (or *entire*) *functions*. Thus, e^z, sin z, cos z, sinh z, cosh z are all integral function.

Note. 2. A function $f(z)$ that can be expressed by means of the operations of addition, subtraction, multiplication, division with polynomials, exponential and logarithmic functions is called an *elementary function*.

2.9. Many-valued Functions. So far we have been considering analytic functions which we defined to be necessarily *one-valued* (or *uniform*). However, several elementary functions such as log z, $\tan^{-1} z$, z^α (α not an integer) etc are *many-valued*, *i.e.*, have more than one value for each value of z. The idea of many-valuedness is very well illustrated by considering the simple function $w=z^{\frac{1}{2}}$ which is two-valued.

If $z=re^{i0}$. the function $w=z^{\frac{1}{2}}$ has the two values

$w_1=\sqrt{r}\ e^{\frac{i0}{2}}$, $w_2=\sqrt{r}\ e^{i\left(\frac{0}{2}+\pi\right)}=-w_1$.

Either of the two functions w_1 and w_2 which $z^{\frac{1}{2}}$ represents is, however, analytic except at $z=0$; and the two functions are called the *branches* of the two-valued function w.

A branch of the many-valued function $f(z)$ is any one-valued analytic function that for each z assumes one of the values of $f(z)$. The requirement of analyticity prevents a branch from taking on a random selection of the values of $f(z)$.

For $0=0$, $w_1=\sqrt{r}$, $w_2=-\sqrt{r}$ and both the branches are one-valued functions of z defined for all values of z. As θ varies from 0 to π, *i.e.*, the variable z describes a circle of radius r about the origin, w_1 varies continuously from $\sqrt{r}$ to $-\sqrt{r}$, *i.e.*, the final value is w_2. If, however, z describes the circle $|z|=r$ a second time, the values of w_1 continue those of w_2 and at the end of the second circuit $w_2=w_1$ along the positive real axis.

If the circuit described by z does not enclose the origin, 0 will regain its initial value and w_1 and w_2 will retain their initial values.

The branch w_1 is clearly discontinuous at points on the positive real axis, since

$$\lim_{0\longrightarrow 2\pi}(\sqrt{r}\,e^{i0/2})=-\sqrt{r}\text{, while }\lim_{0\longrightarrow 0}\sqrt{r}\,e^{i0/2}=\sqrt{r}$$

i.e., the values just below and just above the real axis differ in sign and are not zero (except at the origin itself). Hence, $\frac{dw_1}{dz}$ does not exist when z is real and positive.

Thus, we see that the equation $w=z^{\frac{1}{2}}$ has no continuous one-valued solution defined for the whole complex plane, but it defines a two-valued function of z. Each of the branches is a one-valued function in the z—plane if we make a narrow slit, extending from the origin to infinity along the positive real exis. That part of the real axis for which $x\geqslant 0$ is called a *branch-cut* (or *cross-cut*) for the branch w_1 and the branch is analytic except at points on this cut.

The cut is a boundary introduced so that the corresponding branch is one-valued and analytic throughout the open region bounded by the cut. The open region bounded by the cut is spoken of as the *cut-plane.*

It should be understood here that we chose for convenience the positive real axis as the branch-cut. We might equally well have taken the negative real axis; or, more generally, any Jordan arc extending from $z=0$ to infinity. Since the cut serves to make θ unique, it is clear that every branch cut for the function $z^{\frac{1}{2}}$ must begin at the point $z=0$. *Such an origin of branch cuts is called a branch point of the function.* Since the values of w_1 and w_2 are interchanged as z describes a closed path about the origin, *a branch point may also be defined as that point such that some of the branches of a many-valued function interchange as the independent variable describes a closed path about it.* Since turning about $z=\infty$ means describing a large cricle about the origin, the point $z=\infty$ is also a (conventional) branch point for $w=z^{\frac{1}{2}}$.

The function $w=z^{\frac{1}{2}}$ is two-valued in any region which contains either of the branch points $z=0$ and $z=\infty$, and thus $z^{\frac{1}{2}}$ cannot be analytic throughout these regions. We conclude, therefore, that $z=0$ and $z=\infty$ are singular points of the function $w=z^{\frac{1}{2}}$. A branch point of a function $f(z)$ can thus be defined as a point z_0 such that in no neighbourhood $|z-z_0|<\varepsilon$ is the function $f(z)$ one-valued. If the function $f(z)$ is $n-$valued in a neighbourhood $|z-z_0|<\varepsilon$ of the point z_0, we say that z_0 *is a branch point of order* $(n-1)$ *of* $f(z)$. Thus $z=0$ and $z=\infty$ are branch points of the first order of the function $w=z^{\frac{1}{2}}$.

In the case $w=z^{\frac{1}{2}}$, $z=0$ is a singular point of each of the branches. In general, a branch point may not be a singular point for every branch of a many-valued function but it must be one for some of the branch points.

Note 1. It should be noticed that a branch point is not necessarily an infinity of the function.

Note 2. The regular and singular points of a many-valued function cannot, in general, be defined so simply as in the case of one-valued functions and the particular branches should be considered seperately. A regular point of a branch is defined in the same

way as for a one-valued function. A singularity such as a branch point cannot, however, be assigned to any particular branch.

2·91. The function $w=\sqrt{(z-z_1)(z-z_2)..(z-z_n)}$.

This is a two-valued function whose branch points are $z=z_1$, $z=z_2,\ldots,$ $z=z_n$ if n is even. If n is odd, $z=\infty$ is a branch point in addition.

When n is even, the z—plane is cut along straight lines joining pairs of points z_r, z_s ($r, s=1, 2, 3,\ldots, n$) and when n is odd, one of the points z_n is joined to $z=\infty$. When cuts are made in this manner, each of the branches is analytic on the cut-plane.

Note. When n is even, $z=\infty$ is not a branch point.

2·92. The Logarithmic Function. The logarithm of a complex variable is defined in the same way as the logarithm of a real variable. Thus, log z is defined by the equation $e^w=z$. Unlike the corresponding equation in real variables, this equation in complex variables has an infinite number of solutions and each of these solutions is called a logarithm of z. If $w=u+i\,v$, then

$$e^u(\cos v+i\ \sin v)=z.$$

It follows from this equation that v is one of the values of amp z, and $e^u=|z|$, *i. e.* $u=\log|z|$. Hence, every solution of $e^w=z$ is of the form

$$w=\log|z|+i\ \text{amp}\ z. \qquad \ldots\ldots(1)$$

On writing $z=re^{i\theta}$ and denoting the function w by Log z, the equation (1) becomes

$$\text{Log}\ z=\log r+i\theta, \qquad \ldots\ldots(2)$$

where θ is defined only upto integral multiples of 2π, so that log z is an infinitely many-valued function of z. Each value of θ determines a branch of log z. When the value of θ is restricted to the interval $-\pi<\theta\leqslant\theta$, we get the most important branch of log z which is called its *principal value* and is denoted by *log z*. Thus, if $z=re^{i\theta}$, $-\pi<\theta\leqslant\pi$, then Log $z=\log z+2\,n\pi i$, where n is an integer, positive, negative or zero.

Hence, we can pass from the principal value to any other branch of log z by surrounding the origin a suitable number of

times. The origin is, therefore, a branch point and is known as a *logarithmic branch point.*

The restriction $-\pi<\theta\leqslant\pi$, which determines the principal value of log z, prevents the passing of the point z along the negative real axis. If the z—plane is cut along the real axis from $z=0$ to $z=-\infty$, every branch of the function log z will be an analytic function on this cut plane.

The cut along the negative real axis is a matter of convention as explained in § 2.9, we can place the cut along the positive real axis or along any line from $z=0$ extending upto infinity.

2.93. The Functions a^z, z^a. If a and z denote any complex numbers, provided $a\neq 0$, then the general exponential function a^z is defined by the equation

$$a^z = e^{z\,\mathrm{Log}\,a} = e^{z\,(\log a + 2n\pi i)}.$$

Hence, a^z is an infinitely many-valued function and its principal value is the analytic function given by $e^{z \log a}$, where log a is the principal value of Log a. The other values of a^z corresponding to the different values of Log a are called its *subsidiary values.*

The general power function z^a is defined by the equation

$$z^a = e^{a\,\mathrm{Log}\,z}.$$

If the z-plane is cut along the negative real axis from $z=0$ to $z=-\infty$, every branch of log z is an analytic function on this cut-plane and thus z^a being an analytic function of an analytic function is also analytic on the cut-plane. The principal value of z^a is defined as the analytic function

$$z^a = e^{a \log z},$$

in terms of the principal value of Log z.

Ex. 1. *Show that the values of i^i are all real. Find the principal value.*

We have $i^i = e^{i\,\mathrm{Log}\,i}$

$$= e^{i\,[\log 1 + i\,(\pi/2 + 2n\pi)]}, \quad \text{where } n=0, \pm 1, +2, \ldots\ldots$$

$$= e^{-\pi/2 - 2n\pi}, \quad (n=0, \pm 1, \pm 2\ldots\ldots),$$

which shows that all the values of i^i are real.

Since the principal value of amp $i=\pi/_2$, the principal value of $i^i=e^{-\pi/2}$.

Ex. 2. *If* $w=\sin^{-1}z$, *show that* $w=m\pi\mp i\log\{iz+\sqrt{(1-z^2)}\}$ *according as the integer* m *is even or odd, a branch-cut being made along the real axis from* 1 *to* ∞ *and from* $-\infty$ *to* -1 *to ensure the one-valuedness of the logarithm.*

We have $z=\sin w=\dfrac{1}{2i}(e^{iw}-e^{-iw})$,

i. e. $e^{2iw}-2ize^{iw}-1=0$;

$\therefore$ $e^{iw}=iz\pm\sqrt{1-z^2}$,

and $w=\sin^{-1}z=\dfrac{1}{i}\operatorname{Log}(iz\pm\sqrt{1-z^2})$.

If iw_0 is one of the logarithms of $iz+\sqrt{1-z^2}$, the others are $iw_0+2n\pi i$, where $n=0, \pm1, \pm2,\ldots\ldots$ On the other hand

$iz-\sqrt{1-z^2}=-\dfrac{1}{iz+\sqrt{1-z^2}}$, so that one of its logarithms is $i\pi-iw_0$, and, therefore, the others are $i\pi-iw_0+2n\pi i$.

Hence, $w=w_0+2n\pi$, or $\pi-w_0+2n\pi$,

i. e. $=m\pi\pm\dfrac{1}{i}\log(iz+\sqrt{1-z^2})=m\pi\mp i\log(iz+\sqrt{1-z^2})$,

the sign taken being $-$ or $+$, according as the integer m is even or odd. The many-valued character of w arises because of w involving two many—valued functions, *i. e.*, $\sqrt{1-z^2}$ and the logarithm. The two branch points of $\sqrt{1-z^2}$ are $z=\pm1$. When z describes a closed path about $z=1$ or $z=-1$, the amplitude of $1-z^2$ is increased by 2π, *i. e.*, the two branches of $\sqrt{1-z^2}$ are exchanged. The branches of w, therefore, are exchanged along $(1, \infty)$ and $(-1, -\infty)$ and if the z—plane is cut along the real axis from 1 to ∞ and from $-\infty$ to -1, each of the branches of w is one—valued on the cut—plane.

Exercise 2(b)

1. Determine where the following functions are analytic :

(i) $\tan z=\dfrac{\sin z}{\cos z}$. (ii) $\tanh z=\dfrac{\sinh z}{\cosh z}$.

(iii) $\dfrac{e^z}{z\cos z}$. (iv) $\dfrac{e^z}{\sin z+\cos z}$.

2. If $\cos(\alpha+i\beta)\cosh(x+iy)=1$, where α, β, x, y are all real, prove that, in general, $\tan a\tanh b=\tanh x\tan y$.

3. If $\tanh(u+iv)=x+iy$, where u, v, x, y are real, find x and y in terms of u and v. Find the values of u and v when $x=y=1$.

4. Prove that the conjugates of the complex numbers $\sin z$ and $\cos z$ are $\sin\bar{z}$ and $\cos\bar{z}$ respectively. Hence show that
$|\sin z|^2=\frac{1}{2}(\cosh 2y-\cos 2x)$, $|\cos z|^2=\frac{1}{2}(\cosh 2y+\cos 2x)$.

5. Prove that $\sinh|\operatorname{Im} z| \leqslant \left|\begin{matrix}\sin\\ z\\ \cos\end{matrix}\right| \leqslant \cosh|\operatorname{Im} z|$.
Deduce that $|\sin z|$ and $|\cos z|$ tend to infinity as z tends to infinity in either of the angles $\varepsilon \leqslant \operatorname{amp} z \leqslant \pi-\varepsilon$, $\pi+\varepsilon \leqslant \operatorname{amp} z \leqslant 2\pi-\varepsilon$, where $0<\varepsilon<\frac{\pi}{2}$.

6. Determine all analytic branches of the many-valued functions in the domains given :
(i) $\operatorname{Log} z$, $x<0$; (ii) $z^{1/3}$, $x>0$.

7. Prove that the function $f(z)=1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{\lfloor n}z^n$ is analytic when $|z|<1$ and that its derivative is $\alpha f(z)/(1+z)$. Hence deduce that $f(z)=(1+z)^{\alpha}$.

8. Show that the general solution of the equation $z=\cos w$ is $w=2n\pi\pm i\operatorname{Log}(z+\sqrt{z^2-1})$, $(n=0, \pm1, \pm2, \ldots\ldots)$.
Prove that each of the branches of w is analytic in the $z-$plane cut along the real axis from $-\infty$ to -1 and from 1 to ∞.

9. Show that $\frac{d}{dz}(\sin^{-1}z)=\frac{1}{\sqrt{1-z^2}}$, where that branch of $\frac{1}{\sqrt{1-z^2}}$, is chosen which corresponds to the branch of $\sin^{-1}z$ under consideration.

10. Show that the general solution of the equation $z=\tan w$ is
$$w=\frac{1}{2i}\operatorname{Log}\frac{1+iz}{1-iz}.$$
Prove that each branch of this many-valued function is an analytic function in the z—plane cut along the imaginary axis from $-\infty i$ to $-i$ and from i to ∞i, and that the derivative of each such branch is $\frac{1}{1+z^2}$. Show also that, when $|z|<1$, the principal branch is
$$w=z-\frac{z^3}{3}+\frac{z^5}{5}-\ldots.$$

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11. Obtain the principal value of $(1+i)^i$.

Answers [Exercise 2(a)]

1. (i) The infinite rectangular strip bounded by the real axis and the line parallel to the same at a distance -2π.

(ii) & (iii) Hyperbolas (pairs of straight lines for $a=0$).

(iv) & (v) Lemniscates. (vi) Exterior of the circle with centre at the origin, and radius $\frac{1}{d}$ (vii) The right half-plane including its boundary. (viii) The right bisector of the straight line joining z_1 and z_2. (ix) The interior and boundary of the ellipse with foci a, b and major exis k.

5. (i) $z=0$. (ii) $z=\pm 1$.

9. $v=x^3-3xy^2+c_1$, c_1 a real constant ; $f(z)=iz^3+c_2$, c_2 a complex constant.

10. (i) z^3+3z^2+c ;

(ii) $(1-2i)(\sin z+z^2)+c$; c an arbitrary constant.

12. e^z+c, c a constant.

13. $-iz^3+c$, c a constant.

14. $\frac{1+i}{2}\cot z+c$, c a constant.

15. $\cot\frac{z}{2}+\frac{1}{2}(1-i)$.

16. (i) z^3 ; (ii) $-\frac{1}{z}$; (iii) $(a-i\,b)z^2$; (iv) u does not satisfy Laplace's equation, $\therefore$ $f(z)$ does not exist.

Answers [Exercise 2(b)]

1. The functions are analytic except at the following points :

(i) $\frac{\pi}{2}+n\pi$; (ii) $\frac{\pi}{2}i+n\pi i$; (iii) $0, \frac{\pi}{2}+n\pi$;

(iv) $-\frac{\pi}{4}+n\pi$, where $n=0, \pm 1, \pm 2, \ldots\ldots$.

3. $x=\sinh 2u/(\cosh 2u+\cos 2v)$, $y=\sin 2v/(\cosh 2u+\cos 2v)$; when $x=y=1$, $u=\frac{1}{4}\log_e 5$, $v=\frac{1}{2}(n\pi-\tan^{-1}2)$, where $n=\pm 1$, ± 3, ± 5,..... .

6. (i) $\log r+i\,\theta$, $\frac{\pi}{2}+2n\pi<\theta<\frac{3}{2}\pi+2n\pi$, $n=0, \pm 1, \pm 2,\ldots\ldots$;

(ii) $\sqrt[3]{r}\,e^{i\theta/3}$, $-\frac{\pi}{2}+2n\pi<0<\frac{\pi}{2}+2n\pi$, $n=0, 1, 2$.

11. $e^{-\frac{\pi}{4}+i\log\sqrt{2}}$.

CHAPTER III

CONFORMAL MAPPING

3.1. Conformal Mapping. The relation $w=f(z)$ may be interpreted geometrically and is then called a **transformation** or **mapping.** In the case of a real function of a real variable $y=f(x)$, the relation between x and y can be visualized by means of a graph in the xy-plane. This important idea of a geometrical interpretation of the concepts of real analysis can, to a certain extent, be extended to the case of complex functions of a complex variable $w=f(z)$. However, since z and w are each defined by a pair of real numbers, a geometrical interpretation of the function $w=f(z)$ would need four real numbers, thus making it necessary to use four-dimensional space which is not convenient for geometric visualization. To get over this difficulty, separate planes are used for the two variables, z-plane for the z-points and w-plane for the w-points.

Corresponding points in the two planes are called **images** of each other and the point z is said to be **transformed** into (or **mapped** upon, or **represented** by) the corresponding point or points w by means of the transformation $w=f(z)$. The term 'transformation, or mapping' is really justified only when the correspondence between z-points and w-points is one-to-one, *i.e.*, to each z of a domain D in the z-plane corresponds just one $w=f(z)$ of a domain D' in the w-plane and conversely.

Some simple properties of these mappings are obvious. For example, if $w=f(z)$ is analytic in a domain D of the z-plane, any continuous arc contained in D will be mapped upon an arc of the same nature in the w-plane. Writing $w=v(x, y)+iv(x, y)$, where $z=x+iy$, the map of the arc $z=x(t)+iy(t)$, $t_0 \leqslant t \leqslant t_n$ is the arc $w=u[x(t), y(t)]+iv[x(t), y(t)]$ which is continuous, since u and v are continuous functions of t. Similarly, the map of a differentiable

arc in a domain D_z of the z-plane is also a differentiable arc in the corresponding domain D_w of the w-plane.

When $f(z)$ is analytic, the mapping from D_z to D_w has an additional property of being *conformal*. A mapping from D_z to D_w is called **conformal** if it preserves the angle between two differentiable curves in magnitude as well as in sense of rotation.

Note 1. Some writers use the terms conformal and isogonal in the same sense.

Note 2. If $w=f(z)$ maps a domain D_z conformally on a domain D_w, the function is said to be *simple* in D_z.

3.2. **Theorem.** *If the function $f(z)$ be analytic in a domain D of the z-plane, then the mapping $w=f(z)$ is conformal, provided $f'(z)\neq 0$ in D.*

Let z_o be an interior point of D, and C_1 and C_2 be two differentiable curves passing through z_o. Let the tangents at z_o to C_1 and C_2 respectively make angles α_1, α_2 with the real axis.

Now take the points z_1 and z_2 on C_1 and C_2 respectively, and write $z_1=z_o+re^{i\theta_1}$, $z_2=z_o+re^{i\theta_2}$, so that as $r\to 0$, $\theta_1\to\alpha_1$, $\theta_2\to\alpha_2$. Let the points w_o, w_1, w_2 in the w-plane correspond to the points z_o, z_1, z_2 in the z-plane and let C'_1, C'_2 be the maps of C_1, C_2, so that w_1 and w_2 describe the differentiable curves C'_1 and C'_2 respectively. Further, let $w_1-w_o=\rho_1 e^{i\phi_1}$ and $w_2-w_o=\rho_2 e^{i\phi_2}$, so that $\phi_1\to\beta_1$ as $\rho_1\to 0$ and $\phi_2\to\beta_2$ as $\rho_2\to 0$.

Since $f(z)$ is analytic at z_o, the derivative of the function at z_o

$$f'(z_o)=\underset{z_1\longrightarrow z_o}{Lt}\ \frac{w_1-w_o}{z_1-z_o}=Lt\ \frac{\rho_1\, e^{i\phi_1}}{re^{i\theta}}. \qquad \ldots\ldots\ldots(1)$$

If $f'(z_o)\neq 0$, it can be written in the form $R_o e^{i\psi_o}$, then from (1),

$$Lt\ \frac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}}=R_o e^{i\psi_o}\ . \qquad \ldots\ldots\ldots(2)$$

Hence, $Lt\ (\phi_1-\theta_1)=\psi_o$, *i.e.*, $\beta_1=\alpha_1+\psi_o$. $\ldots\ldots\ldots(3)$

Thus, the tangent to the curve C'_1 at w_o makes an angle $\alpha_1+\psi_o$ with the real axis, *i.e.*, the directed tangent to the curve C_1 is rotated through the angle $\psi_o=\text{amp}\, f'(z_o)$.

Similarly, the tangent to C'_2 at w_o makes an angle $\beta_2=\alpha_2+\psi_o$.....(4) with the real axis. From (3) and (4). it follows that $\beta_2-\beta_1=\alpha_2-\alpha_1$.

Consequently, the angle at z_o from C_1 to C_2 is the same, both in magnitude and sense as the angle at w_o from C'_1 to C'_2, provided $f'(z_o)\neq 0$; *i.e.*, the mapping is conformal.

A point at which $f'(z)=0$ is called a **critical** point of the mapping. We discuss below this case in § 3.22.

3.21. **Geometrical Interpretation.** From equation (2) above, we have $Lt\ \frac{\rho_1}{r}=R_o$, or $\frac{\rho_1}{r}=R_o+\varepsilon$, where $\varepsilon\to 0$ as $z_1\to z_o$.

Hence, in a neighbourhood of the point z_o, the ratio of the magnitude of an element of the resulting configuration in the w-plane to the magnitude of the corresponding element of the given configuration in the z-plane is approximately $R_o=|f'(z_o)|$, which may be called the *ratio* (or *coefficient*) of *linear magnification*. Since R_o is constant for z_o, *i.e.*, the magnification is the same in all directions through the point z_o, we conclude that the image of each small figure near the point z_o **conforms** to the original figure in the sense that it has approximately the same shape. Large figures may, however, be mapped upon figures totally different from the original, since the coefficient of magnification and the angle of rotation both vary from point to point.

3.22. **The case** $f'(z)=0$. If the first $(n-1)$ derivatives of the function $f(z)$ vanish at z_o, while $f^n(z_o)\neq 0$, then in the neighbourhood of z_o, we have, by Taylor's theorem (see § 4.5)

$$f(z)=f(z_0)+\sum_{k=n}^{\infty}\frac{f^k(z_0)}{\lfloor k}(z-z_0)^k.$$

Hence, $f(z_1)-f(z_0)=a_n(z_1-z_0)^n+\ldots$,

where $a_n\frac{f^n(z_0)}{\lfloor n}\neq 0$;

or $w_1-w_0=a_n(z_1-z_0)^n+\ldots$,

or $\rho_1 e^{i\phi_1}=|a_n|r^n e^{i(n\theta_1+\lambda)}$, where $\lambda=\text{amp } a_n$.

Taking limits, we have $\beta_1 = Lt \ \ \phi_1 = n\alpha_1 + \lambda$. Similarly, $\beta_2 = n\alpha_2 + \lambda$.

Thus, the angle between the differentiable curves C'_1 and C'_2 at w_0 is $\beta_2 - \beta_1 = n(\alpha_2 - \alpha_1)$, *i.e.*, it is magnified n times by the mapping $w = f(z)$.

The coefficient of magnification, $R_0 = \text{Lt} \frac{\rho_1}{r}$ is obviously zero in this case.

Hence, the conformal property does not hold at a point, where $f'(z) = 0$.

Note. If we discuss the mapping from the w-plane to the z-plane, we see that the mapping ceases to be conformal if $\frac{dz}{dw} = 0$, *i.e.*, $f'(z) = \infty$. Thus, a point at which $f'(z) = \infty$ is also called a **critical** point of the mapping.

3.23. Isogonal Mapping. A mapping which preserves the magnitudes of the angles but not necessarily their sense is called **isogonal**. For example, the mapping $w = \bar{z}$ transforms every point into its reflection in the real axis, *i.e.*, it preserves the magnitude of the angles but not their sense. Hence, the mapping is isogonal but not conformal.

In general, the mapping $w = f(\bar{z})$, where $f(z)$ is an analytic function of z, is isogonal but not conformal ; for, it combines the two mappings

(*i*) $\zeta = \bar{z}$, and (*ii*) $w = (\zeta)$.

In (i) the angles preserve their magnitude but not their sense, and in (ii) the angles preserve their magnitude as well as their sense. Hence, in the resulting mapping, the angles preserve their magnitude but not their sense.

3.3. The Converse Problem. We have seen (§ 3.2) that if $f(z)$ be analytic in a domain D of the z-plane and $f'(z) \neq 0$ in D, then the mapping $w = f(z)$ is conformal.

Since $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix}$ [$\because$ the function $f(z)$ is analytic]

$$= u_x^2 + v_x^2 = |f'(z)|^2 \neq 0,$$

it follows that the mapping from the (x, y)-plane to (u, v)-plane is conformal provided $u_x v_y - v_x u_y$, the Jacobian of the transformation, does not vanish anywhere in D.

We now consider the converse of this theorem.

Theorem. *The mapping defined by the pair of real fuuctions*

$$u=u(x, y),\ v=v(x, y)$$

which possess continuous partial derivatives of the first order in a domain D_z of the z-plane and $u_x v_y - v_x u_y \neq 0$, is conformal only if $w=u+iv$ is an analytic function of $x+iy$ in D_z.

Let $z=x_1(t)+iy_1(t)$ and $z=x_2(t)+iy_2(t)$ be two differentiable curves C_1 and C_2 in D_z intersecting in a point $z_0=(x_0, y_0)$ of D_z and let the tangents to C_1 and C_2 at the point z_0 make angles α_1 and α_2 respectively with the real axis, then

$$\tan \alpha_1 = \frac{y'_1(t_0)}{x'_1(t_0)},\quad \tan \alpha_2 = \frac{y'_2(t_0)}{x'_2(t_0)},$$

where t_0 is the value of t corresponding to the point z_0. Further, let $\alpha=\alpha_2-\alpha_1$.

The discussion remains quite general, if we suppose that the tangent at z_0 to C_1 is parallel to the real axis, *i.e.*, $y'_1=0$, so that $\alpha_1=0$ and $\alpha=\alpha_2$.

Under the given transformation, C'_1 and C'_2, the maps of C_1 and C_2 respectively, are given by

$$w=u_1(t)+iv_1(t)=u_1\{x_1(t),\ y_1(t)\}+iv_1\{x_1(t), y_1(t)\},$$

and $$w=u_2(t)+iv_2(t)=u_2\{x_2(t), y_2(t)\}+iv_2\{x_2(t), y_2(t)\}$$

in the corresponding domain D_w of the w-plane.

Let the tangents to C'_1 and C'_2 at the point w_o corresponding to the point z_0 make angles β_1 and β_2 respectively with the real axis; then, since u and v possess continuous partial derivatives of the first order,

$$\tan \beta_1 = \frac{v'_1}{u'_1} = \frac{v_x x'_1 + v_y y'_1}{u_x x'_1 + u_y y'_1} = \frac{v_x}{u_x} \qquad (\because\ y'_1=0),$$

$$\text{and } \tan \beta_2 = \frac{v'_2}{u'_2} = \frac{v_x x'_2 + v_y y'_2}{u_x x'_2 + u_y y'_2} = \frac{v_x + v_y \tan \alpha}{u_x + u_y \tan \alpha}$$

$$\left(\because \tan \alpha = \frac{y'_2}{x'_2} \right),$$

where all the derivatives have their values at (x_0, y_0).

Now, if the mapping from D_z to D_w is conformal, $\beta_2-\beta_2=\alpha$,

$$i.e., \quad \tan\alpha=\tan(\beta_2-\beta_1)=\frac{\tan\beta_2-\tan\beta_1}{1+\tan\beta_2\tan\beta_1}$$

$$=\left(\frac{v_x+v_y\tan\alpha}{u_x+u_y\tan\alpha}-\frac{v_x}{u_x}\right)\Big/\left[1+\frac{v_x}{u_x}\cdot\frac{v_x+v_y\tan\alpha}{u_x+u_y\tan\alpha}\right]$$

$$=\frac{(u_xv_y-v_xu_y)\tan\alpha}{u^2{}_x+v^2{}_x+(u_xu_y+v_xv_y)\tan\alpha},$$

or $(u_xu_y+v_xv_y)\tan\alpha+(u^2{}_x+v^2{}_x-u_xv_y+v_xu_y)=0.$

Since this equation is satisfied for all values of α in the interval $0<\alpha<\frac{\pi}{2}$, it follows that it is an identity ;

$$\therefore \quad u_xu_y+v_xv_y=0,\ u^2{}_x+v^2{}_x=u_xv_y-v_xu_y. \qquad \ldots\ldots\ldots(1)$$

These relations can also be written as two homogeneous linear equations in (u_x-v_y) and (u_y+v_x) as follows :

$$\left.\begin{aligned}-v_x(u_x-v_y)+u_x(u_y+v_x)=0,\\ u_x(u_x-v_y)+v_x(u_y+v_x)=0.\end{aligned}\right\} \qquad \ldots\ldots\ldots(2)$$

These equations have a non-zero solution, only if

$$\begin{vmatrix}-v_x & u_z\\ u_x & v_x\end{vmatrix}=0, \quad i.e., \text{ if } u_x{}^2+v_x{}^2=0. \qquad \ldots\ldots\ldots(3)$$

Now (3) cannot hold, since, by (1), we have $u_xv_y-v_xu_y=0$, which shows that the Jacobian of the transformation vanishes.

Hence, the equations (2) can only have the solution

$$u_x-v_y=0, \quad \text{and} \quad u_y+v_x=0, \quad \text{or} \quad u_x=v_y,\ u_y=-v_z,$$

i.e., at a point where the mapping from (x, y)-plane to (u, v)-plane is conformal and the Jacobian of the transformation does not vanish, the Cauchy-Riemann conditions are satisfied.

Hence, $w=u(x, y)+iv(x, y)$ is an analytic function of $z=x+iy$ in D_z.

Ex. 1. *Deduce from the principal of conformal mapping the theorem that the curves $u=$constant, $v=$constant intersect at right angles where u and v are conjugate functions.* (*Agra 1953*)

Since u and v are conjugate function, the function.

$$w=u+iv=f(z)$$

is analytic in a domain D_z, say, of the z-plane. Hence, the function $w=f(z)$ maps D_z conformally on D_w, the corresponding domain of w-plane, if $f'(z)\neq 0$ in D_z.

Let $u=c_1$ and $v=c_2$, where c_2 are constants, meet at w_o in D_w and let z_o be the corresponding point in z_o. Now, the straight lines $u=c_1$ and $v=c_2$ in the w-plane intersect at right angles; hence, their images through z_o in D_z must intersect at right angles, since the mapping is conformal.

Ex. 2. *Show that, if the function* $w=f(z)$, *analytic in* $|z|=R$, *maps the circle* $|z|=r<R$ *on a rugular are* C *in the* w*-plane, then the length of* C *is given by*

$$\int_0^{2\pi} |f'(re^{i\theta})|\ rd\,0.$$

Let an element of the arc δs of the circle $|z|=r$ be mapped on the corresponding element of the arc $\delta\sigma$ of C; then, since $w=f(z)$ is analytic in $|z|=R$, the mapping is conformal and, therefore, $\delta\sigma=|f'(z)|\,\delta s$.

$$\text{Hence, lenghth of } C=\int_0^{2\pi} |f'(re^{i\theta})|\,.\,rd\theta,$$

where $z=re^{i\theta}$ $(0\leqslant 0<2\pi)$, and $ds=\sqrt{\overline{(rd\theta)^2+(dr)^2}}=rd\theta(\because dr=0)$.

3.4. **Simple Transformations.** If $w=\alpha z+\beta$, the transformation is called **linear**; if $w=\dfrac{\phi(z)}{\psi(z)}$, where $\phi(z)$ and $\psi(z)$ are polynomials, the transformation is said to be *rational.* A transformation of the type $w=\dfrac{\alpha z+\beta}{\gamma z+\delta}$ is known as **bilinear** or *linear fractional.* The geometrical meaning of linear and bilinear transformations will now be investigated. To justify the use of such graphic terms as *translation*, *rotation* and *stretching*, we shall consider the mapping as effected in one plane.

3.41. **The Transformation** $\mathbf{w=z+\beta}$. From the geometric interpretation of addition it will be seen that the z-points are trans-

formed into the corresponding w-points by moving each z-point in the direction of the vector from the origin to the point β and to a distance equal to $|\beta|$. The motion just described is called *translation*. It is obvious that a translation of the complex plane does not change the shape and size of any configuration and, therefore, transforms any curve into a congruent curve.

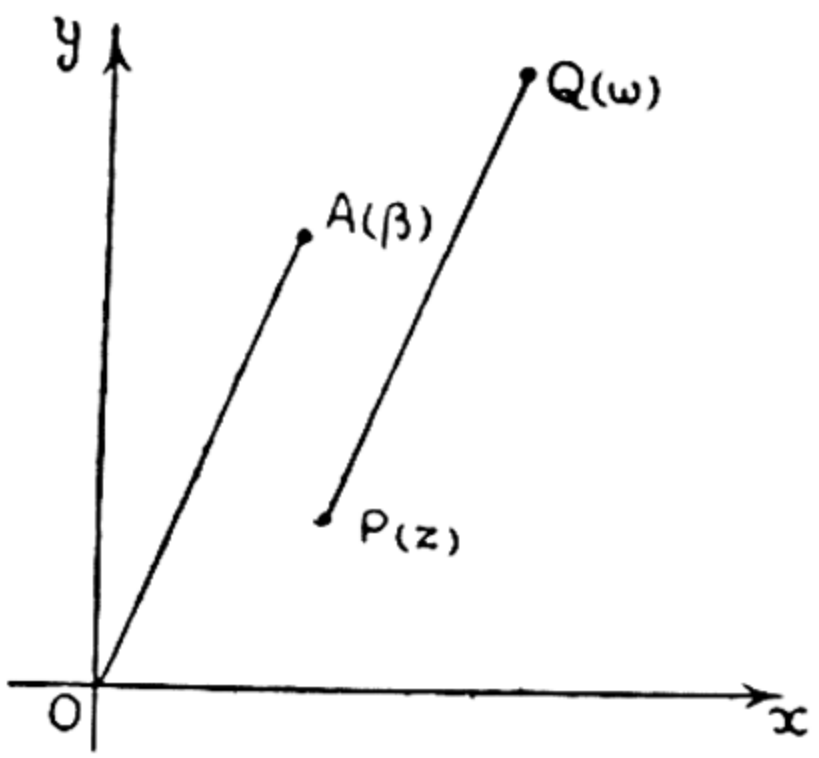


Fig. 10

3.42. The Transformation $w=\alpha z$. This transformation gives $|w| = |\alpha| \cdot |z|$, and amp w=amp α+amp z.

Consequently, to derive the point w corresponding to the point z, we multiply the modulus of z by $|\alpha|$ and then *rotate* (or, *turn*) the *magnified* vector through an angle amp α in the counterclockwise direction. Multiplication of $|z|$ by $|\alpha|$ is also called an *expansion* (or *magnification*, or *stretching*) or *contraction*. We can also derive the point w by first rotating the vector from the origin to the point z through an angle amp α and then multiplying it by $|\alpha|$. This transformation also leaves the form of any configuration unchanged.

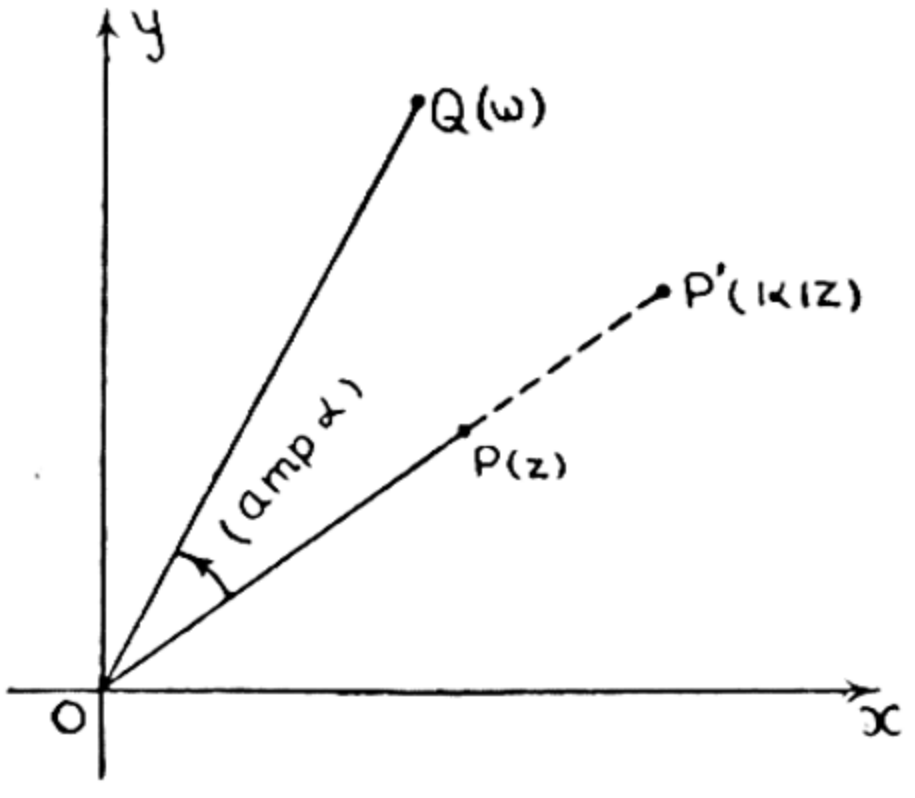


Fig. 11

3.43. The Transformation $w=\alpha z+\beta$. The geometrical significance is easy to observe if we regard this transformation as a combination of the two preceding transformations. The relation $w=\alpha z+\beta$ is obtained by introducing lhe auxiliary variable $z'=\alpha z$, ...(1) and then putting $w=z'+\beta$.....(2). The same relation may also be obtained by first introducing the auxiliary variable $z''=z+\beta/\alpha$, ...(3) and then putting $w=\alpha z''$.(4).

The amount of rotation and magnification in (1) and (4) is the same. The translations defined by (2) and (3) are however different. Thus, the process of rotation and magnification and the process of translation are not interchangeable processes.

Under the transformation $w=\alpha z+\beta$, there is one invariant point $\frac{\beta}{1-\alpha}$; for, since the z-point is identical with the corresponding w-point, $z=\alpha z+\beta$, or $z=\frac{\beta}{1-\alpha}$, if $\alpha\neq 1$ and the given transformation represents rotation and magnification about this invariant point as can be easily seen.

For example, the transformation $w=(1+i)\ z+1-2i$ maps the rectangular region in the z-plane into the rectangular region shown in the w-plane. This is evidently geometrically,

since amp $(1+i)=\frac{\pi}{4}$,

and $|1+i|=\sqrt{2}$.

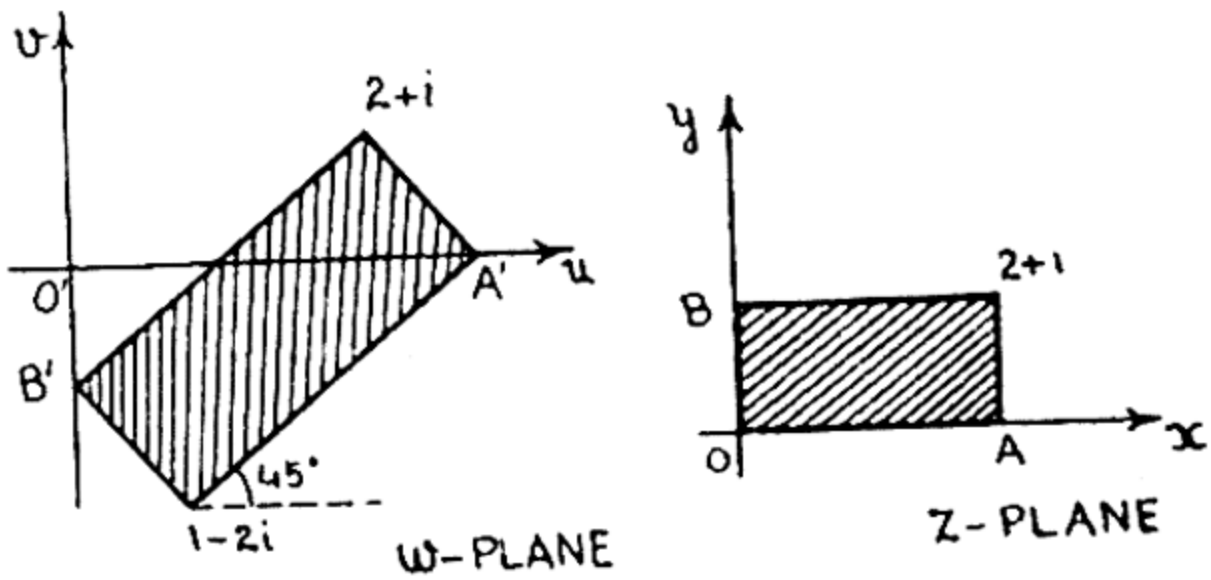


Fig. 12

The point $2+i$ is the invariant point; for $\frac{\beta}{1-\alpha}=\frac{1-2i}{1-(1+i)}$ $=2+i$.

The rotation is about this point.

3.44. **The Transformation $w=\frac{1}{z}$.** Putting $w=\rho e^{i\phi}$ and $z=re^{i0}$, this transformation becomes $\rho e^{i\phi}=\frac{1}{r}e^{-i0}$. By introducing the auxiliary variable w', the transformation $w=\frac{1}{z}$ can be regarded as

a combination of two transformations $w'=\frac{1}{r}e^{i\theta}$, $w=\bar{w}'$. The first is an *inversion* w.r.t. the unit circle $|z|=1$, and the second is a *reflection* in the real axis. In the figure if P is the z-point, then the corresponding w-point is Q ; w' is represented by P′, where $OP.OP'=1$.

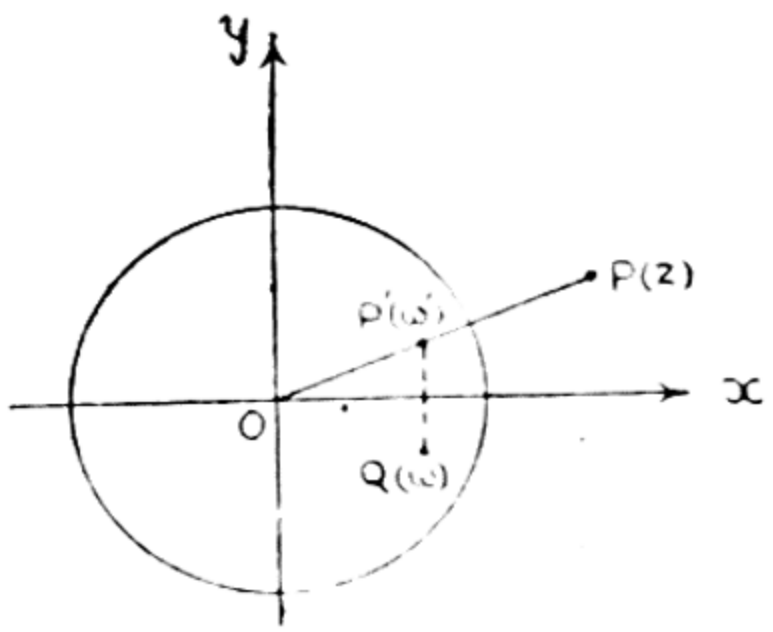


Fig. 13.

In cartesian coordinates, $w=u+iv=\frac{1}{x+iy}$, which gives the relations $u=\frac{x}{x^2+y^2}$, $v=\frac{-y}{x^2+y^2}$; and $x=\frac{u}{u^2+v^2}$, $y=\frac{-v}{u^2+v^2}$.

Regarding a straight line as a limiting case of a circle whose radius tends to infinity, we see that the circle

$$A(x^2+y^2)+Bx+Cy+D=0, \qquad \ldots\ldots(1)$$

is transformed into

$$D(u^2+v^2)+Bu-Cv+A=0, \qquad \ldots\ldots(2)$$

which is also a circle.

Hence, the transformation $w=\frac{1}{z}$ maps circles on circles, straight lines being regarded as degenerate circles.

It is known from geometry that the angle of intersection of two curves is preserved by inversion w.r.t. a circle but the sense is changed. The reflection in the real axis also preserves the magnitude of the angles but not the sense. It follows, therefore, that the mapping $w=\frac{1}{z}$ is conformal except at $z=0$.

Note. This transformation is also called a *reciprocation.*

3.45. The Point at Infinity. Since the transformation $w=\frac{1}{z}$ carries points outside the circle $|z|=R$ into points inside the

circle $|w| = \frac{1}{R}$, it follows that the point $w=0$ is not the image of any point in the finite z-plane. However, by making R sufficiently large, all the points outside the large circle $|z| = R$ map on the interior of the arbitrarily small circle $|w| = \frac{1}{R}$. Hence, by making $R \longrightarrow \infty$, we see that the *point at infinity* or the *infinite point* $z=\infty$ is, formally the image of the point $w=0$ under the mapping $w = \frac{1}{z}$.

3.46. The Bilinear Transformation $w = \frac{\alpha z+\beta}{\gamma z+\delta}$.

This transformation can be written as

$$w = \frac{\alpha}{\gamma} + \frac{\beta\gamma-\alpha\delta}{\gamma} \cdot \frac{1}{\gamma z+\delta},$$

where $\gamma \neq 0$, and can therefore be effected by combining the three transformations

$$z_1 = \gamma z+\delta, \ \ldots(1) \quad z_2 = \frac{1}{z_1}, \ \ldots(2) \text{ and } w = \frac{\alpha}{\gamma} + \frac{\beta\gamma-\alpha\delta}{\gamma} z_2 \ \ldots(3)$$

If $\gamma=0$, the given transformation reduces to the linear transformation of § 3.43. The expression $\alpha\delta-\beta\gamma$ or $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$, called the *determinant* of the transformation, must not vanish. If $\alpha\delta-\beta\gamma=0$, the given relation between w and z would reduce to $w = \frac{\alpha}{\gamma}$, which is either a constant or meaningless.

Thus, when $\gamma \neq 0$ and $\alpha\delta-\beta\gamma \neq 0$, the transformations (1) and (3) are linear while (2) is a reciprocation. Hence, the given transformation is conformal and maps circles on circles with lines as limiting cases except when $z=-\delta/\gamma$ which is the only finite singularity of w. Since the image of $z=-\delta/\gamma$ is $w=\infty$, circles passing through $z=-\delta/\gamma$ are mapped upon straight lines in the w-plane.

The transformation inverse to $w=\frac{\alpha z+\beta}{\gamma z+\delta}$, $\alpha\delta-\beta\gamma\neq 0$; is $z=\frac{-\delta w+\beta}{\gamma w-\alpha}$, $(-\delta)(-\alpha)-\beta\gamma\neq 0$; which is also a bilinear transformation.

Since the function $w=\frac{\alpha z+\beta}{\gamma z+\delta}$ is analytic for all values of z except $z=-\delta/\gamma$ and since the derivative $\frac{dw}{dz}=\frac{\alpha\delta-\beta\gamma}{(\gamma z+\delta)^2}$ is zero or infinite when $z=\infty$ or $z=-\delta/\gamma$, it follows that the bilinear transformation is conformal at all points except $z=\infty$ and $z=-\delta/\gamma$ which are the critical points of the transformation. However, if we extend the definition of conformal mapping as below, the two critical points no longer remain exceptional.

The function $w=f(z)$ is said to map conformally the neighbourhood of $z=\infty$ upon the neighbourhood of w_o, if the function $w=F(\zeta)=f(1/\zeta)$ maps conformally the neighbourhood of $\zeta=0$ upon the neighbourhood of w_o. Here w_o may have the value infinity. A similar extension of definition can be given for the conformal mapping of the neighbourhood of z_o upon the neighbourhood of $w=\infty$.

With these extensions of definition, the mapping is conformal at $z=\infty$ and $z=-\delta/\gamma$. Also, under this transformation the correspondence of the whole closed z-plane to the whole closed w-plane is one-to-one.

Note. This transformation was first studied by Möbius and is also known as a **Möbius Transformation.**

3.47. General Properties of the Bilinear Transformation.

I. *This transformation leaves two points in the complex plane invariant.*

If the bilinear transformation maps any point upon itself, then

$$z=\frac{\alpha z+\beta}{\gamma z+\delta},\ i.e.,\ \gamma z^2+(\delta-\alpha)z-\beta=0,$$

which has two roots, namely $z_1=\frac{\alpha-\delta+\sqrt{(\alpha-\delta)^2+4\beta\gamma}}{2\gamma}$,

and $z_2=\frac{\alpha-\delta-\sqrt{(\alpha-\delta)^2+4\beta\gamma}}{2\gamma}$

The points z_1 and z_2 are the invariant points.

II. *There is always one and only one bilinear transformation which maps any three distinct points upon three desired distinct points.*

The bilinear transformation contains four constant $\alpha.\beta,\gamma,\delta$; but as we may divide both numerator and dominator by one of these without affecting the transformation, we have only three independent constants. Therefore it is to be expected that the transformation can be made to map any three distinct points z_1, z_2, z_3 upon any three desired points w_1, w_2, w_3, that are distinct. Substitution of the corresponding values of z and w in the transformation gives three

relations $w_k=\frac{\alpha z_k+\beta}{\gamma z_k+\delta}$, $k=1, 2, 3,$

i.e., $\gamma w_k z_k+\delta w_k-\alpha z_k-\beta=0, k=1,2,3,$(1)

which are linear homogeneous equations in the four unknowns $\alpha, \beta, \gamma, \delta$, or linear equations in the ratios of some three of $\alpha, \beta, \gamma, \delta$ to the fourth. The simultaneous equations (1) cannot have more than one solution except $\alpha=\beta=\gamma=\delta=0$ in which case the transformation is meaningless.

As $w_r-w_s=\frac{(\alpha\delta-\beta\gamma)(z_r-z_s)}{(\gamma z_r+\delta)(\gamma z_s+\delta)}$, it can be easily seen that the mapping is made by the transformation

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}. \qquad \text{.....(2)}$$

For, $w=w_1$ corresponds to $z=z_1$ and $w=w_3$ corresponds to $z=z_3$. If $z=z_2$, the equation (2) reduces to the form $(w-w_1)(w_2-w_3)=(w-w_3)(w_2-w_2)$ which has the solution $w=w_2$.

Since the transformation (2) is bilinear in z and w, it is the transformation that maps the three given points in the z-plane into the three prescribed points in the w-plane. There cannot be more

than one such bilinear transformation ; for, the simultaneous equations (1) cannot have more than one solution.

III. *The necessary and sufficient condition that any four distinct points* z_1, z_2, z_3, z_4, *of the z-plane may be mapped by the bilinear transformation* $w=\frac{\alpha z+\beta}{\gamma z+\delta}$ *upon any other four distinct points* w_1, w_2, w_3, w_4 *of the w-plane is that*

$$\frac{(w_1-w_2)\,(w_3-w_4)}{(w_1-w_4)\,(w_2-w_3)}=\frac{(z_1-z_2)\,(z_3-z_4)}{(z_1-z_4)\,(z_2-z_3)}.$$

We have the four relations $w_k=\frac{\alpha z_k+\beta}{\gamma z_k+\delta}$, $k=1, 2, 3, 4$;

or $\gamma w_k z_k+\delta w_k-\alpha z_k-\beta=0$, $(k=1, 2, 3, 4)$.(3)

The necessary and sufficient condition that these four equations have a solution other than $\alpha=\beta=\gamma=\delta=0$ is that

$$\begin{vmatrix} w_1z_1 & w_1 & -z_1 & -1 \\ w_2z_2 & w_2 & -z_2 & -1 \\ w_3z_3 & w_3 & -z_3 & -1 \\ w_4z_4 & w_4 & -z_4 & -1 \end{vmatrix}=0.$$

Expanding this determinant in terms of the last two columns by Laplace's development, we have

$$\begin{vmatrix} z_1 & 1 \\ z_2 & 1 \end{vmatrix}\begin{vmatrix} z_3 & 1 \\ z_4 & 1 \end{vmatrix} w_3w_4-\begin{vmatrix} z_1 & 1 \\ z_3 & 1 \end{vmatrix}\begin{vmatrix} z_2 & 1 \\ z_4 & 1 \end{vmatrix} w_2w_4+\begin{vmatrix} z_1 & 1 \\ z_4 & 1 \end{vmatrix}\begin{vmatrix} z_2 & 1 \\ z_3 & 1 \end{vmatrix}\times$$
$$w_2w_3+\begin{vmatrix} z_3 & 1 \\ z_4 & 1 \end{vmatrix}\begin{vmatrix} z_1 & 1 \\ z_2 & 1 \end{vmatrix} w_1w_2-\begin{vmatrix} z_2 & 1 \\ z_3 & 1 \end{vmatrix}\begin{vmatrix} z_1 & 1 \\ z_3 & 1 \end{vmatrix} w_1w_3+$$
$$\begin{vmatrix} z_1 & 1 \\ z_3 & 1 \end{vmatrix}\begin{vmatrix} z_1 & 1 \\ z_4 & 1 \end{vmatrix} w_1w_4=0. \qquad(4)$$

Since $\begin{vmatrix} z_1 & 1 \\ z_2 & 1 \end{vmatrix}\begin{vmatrix} z_3 & 1 \\ z_4 & 1 \end{vmatrix}-\begin{vmatrix} z_1 & 1 \\ z_3 & 1 \end{vmatrix}\begin{vmatrix} z_2 & 1 \\ z_4 & 1 \end{vmatrix}+\begin{vmatrix} z_1 & 1 \\ z_4 & 1 \end{vmatrix}\begin{vmatrix} z_2 & 1 \\ z_4 & 1 \end{vmatrix}\equiv 0,$

we may write the equation (4) in the form

$$(z_1-z_2)\,(z_3-z_4)\{(w_3w_4+w_1w_2)-(w_2w_4+w_1w_3)\}$$
$$+(z_1-z_4)(z_2-z_3)\{(w_2w_3+w_1w_4)-(w_2w_4+w_1w_3)\}=0,$$

or $(z_1-z_2)\,(z_3-z_4)\,(w_1-w_4)\,(w_2-w_3)$
$=(z_1-z_4)\,(z_2-z_3)(w_1-w_2)(w_3-w_4)$;

$$\therefore \quad \frac{(w_1-w_2)\,(w_3-w_4)}{(w_1-w_4)\,(w_2-w_3)}=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_2-z_3)}. \qquad(5)$$

The R.H.S. of (5) is the *cross-ratio* of the four points z_1, z_2, z_3, z_4 and so we have the result that the cross-ratio is invariant for the bilinear transformation.

IV. *Successive bilinear transformations are equivalent to a single bilinear transformation.*

If $w=\dfrac{\alpha z'+\beta}{\gamma z'+\delta}$, $\alpha\delta-\beta\gamma\neq 0$; and $z'=\dfrac{\alpha' z+\beta'}{\gamma' z+\delta'}$, $\alpha'\delta'-\beta'\gamma'\neq 0$

the result is a third bilinear transformation $w=\dfrac{az+b}{cz+d}$,

where $ad-bc=(\alpha\delta-\beta\gamma)(\alpha'\delta'-\beta'\gamma')\neq 0$.

V. *Inverse points w.r.t. a given circle are transformed by the bilinear transformation into inverse points w.r.t. the transformed circle.*

Thus, if the points z' and z'' are inverse *w.r.t.* the circle

$$|z-z_o|=r, \qquad \ldots\ldots(6)$$

we can write $z'=z_o+le^{i\phi}$, $z''=z_o+\dfrac{r^2}{l}e^{i\phi}$, where l is the distance from the point z' to the centre z_o, as shown in the figure.

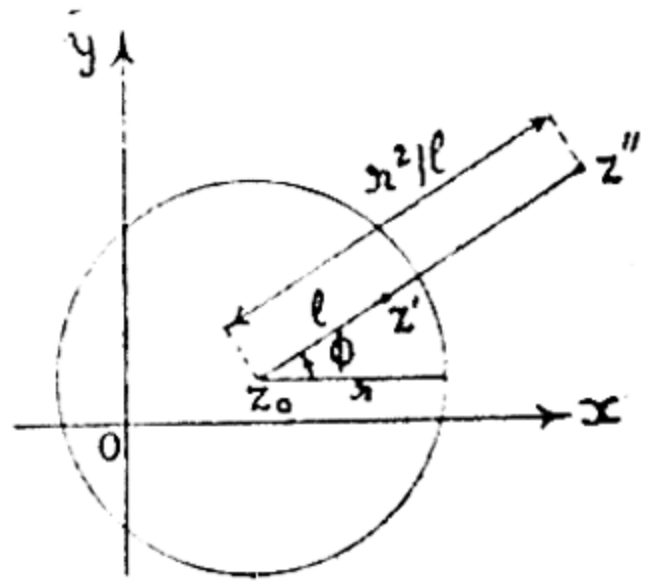


Fig. 14

Now $z=z_o+re^{i\theta}$ is any point of the circle (6),

$$\therefore \quad \left|\frac{z-z'}{z-z''}\right| = \left|\frac{re^{i\theta}-le^{i\phi}}{re^{i\theta}-\frac{r^2}{l}e^{i\phi}}\right|$$

$$=\frac{l}{r}\left|\frac{re^{i\theta}-le^{i\phi}}{le^{i\theta}-rei\phi}\right|=\frac{l}{r}. \qquad \ldots\ldots(7)$$

This is, therefore, the form of the equation of the circle when z' and z'' are inverse points w.r.t. it. Taking $\dfrac{l}{r}=k\neq 1$, the equation (7) becomes

$$\left|\frac{z-z'}{z-z''}\right|=k,\ k\neq 1. \qquad \ldots\ldots(8)$$

This equation gives $|z-z'|^2-k^2|z-z''|^2=0$,

or $(z-z')(\bar{z}-\bar{z}')-k^2(z-z'')(\bar{z}-\bar{z}'')=0$,

$$\text{or} \quad |z|^2-2\mathrm{Re}(z\bar{z}')+|z'|^2-k^2\{|z|^2-2\mathrm{Re}(z\bar{z}'')+|z''|^2\}=0,$$

$$\text{or} \quad |z|^2-2\,\frac{\mathrm{Re}\{(\bar{z}'-k^2\bar{z}'')z\}}{1-k^2}+\frac{|z'|^2-k^2|z''|^2}{1-k^2}=0,$$

$$\text{or} \quad \left|z-\frac{z'-k^2z''}{1-k^2}\right|^2=\frac{|z'-k^2z''|^2}{(1-k^2)^2}-\frac{|z'|^2-k^2|z''|^2}{1-k^2}$$

$$=\frac{|z'-k^2z''|^2-(1-k^2)(|z'|^2-k^2|z''|^2)}{(1-k^2)^2}$$

$$=\frac{k^2|z'-z''|^2}{(1-k^2)^2},$$

i. e., $\left|z-\frac{z'-k^2z''}{1-k^2}\right|=\frac{k|z'-z''|}{|1-k^2|}$, which represents a circle

with centre $z_o=\frac{z'-k^2z''}{1-k^2}$ and radius $=\frac{k|z'-z''|}{|1-k^2|}=r$, say.

Also $|z'-z_o|=\frac{k^2(z''-z')}{1-k^2}$, $|z''-z_o|=\frac{z''-z'}{1-k^2}$;

so that $\frac{z'-z_o}{z''-z_o}=k^2$, a real positive number,

and $|z'-z_o|\,|z''-z_o|=\frac{k^2|z'-z''|^2}{(1-k^2)^2}=r^2$.

Thus, the points z' and z'' are inverse points w.r.t. the circle (8). Hence, the equation (8) necessarily represents a circle w.r.t. which z' and z'' are inverse points.

If $k=1$, the equation (8) represents the right bisector of the straight line joining z' and z'' and points z', z'' are symmetrical about this locus.

Under the transformation $w=\frac{\alpha z+\beta}{\gamma z+\delta}$, the circle (8) is transformed into

$$\left| \frac{\delta w-\beta-z'(-\gamma w+\alpha)}{\delta w-\beta-z''(-\gamma w+\alpha)} \right| = k, \left[\because \quad z = \frac{\delta w-\beta}{-\gamma w+\alpha} \right]$$

or $$\left| \frac{w-\dfrac{\alpha z'+\beta}{\gamma z'+\delta}}{w-\dfrac{\alpha z''+\beta}{\gamma z''+\delta}} \right| = k \left| \frac{\gamma z''+\delta}{\gamma z'+\delta} \right|,$$

or $$\left| \frac{w-w'}{w-w''} \right| = k \left| \frac{\gamma z''+\delta}{\gamma z'+\delta} \right| \neq 1. \qquad \text{.....(9)}$$

where w' and w'' correspond respectively to z' and z''.

Hence, the circle (8) is transformed into the circle (9) with w', w'' as the inverse points w.r.t. it.

Note. The circle (8) is called the *circle of Apollonius*.

Ex. 1. *Find the condition that the transformation* $w=\dfrac{az+b}{cz+d}$ *transforms the unit circle in the w-plane into a straight line in the z-plane.* (*Agra, 1940*)

A straight line in the z-plane is represented by $\left| \dfrac{z-z'}{z-z''} \right| = 1$.

Putting $z'=-\dfrac{b}{a}$, $z''=-\dfrac{d}{c}$, this line is transformed into

$$\left| \frac{z+\dfrac{b}{a}}{z+\dfrac{d}{c}} \right| = 1, \text{ or } |w| = \left| \frac{a}{c} \right|, \text{ since } w = \frac{a}{c} \cdot \frac{z+\dfrac{b}{a}}{z+d\dfrac{d}{c}}$$

which will be a unit circle if $|a| = |c|$.

Note. Since $\left| \dfrac{z+\dfrac{b}{a}}{z+\dfrac{d}{c}} \right| = k(\neq 1)$ represents a circle in the z-plane, the circle in the w-plane corresponding to it is obviously

$$|w| = k \left| \frac{a}{c} \right|.$$

By taking particular value of a, b, c, d and k, the boundaries in the w-plane corresponding to given boundaries in the z-plane can be easily determined.

For example, by taking $\frac{b}{a}=-i$, $\frac{d}{c}=i$, $k=1$, we can make $\left|\frac{z-i}{z+i}\right|=1$, *i.e.*, the real axis in the z-plane correspond to the circle $|w|=1$ in the w-plane, provided $|a|=|c|$.

Ex. 2. *Find the bilinear transformation which transforms the half-plane* $\mathrm{Re}(z)\geqslant 0$ *into the unit circle* $|w|\leqslant 1$.

To points $z, -\bar{z}$ symmetrical about $\mathrm{Re}(z)=0$ (*i.e.*, the imaginary axis) correspond points $w, \frac{1}{\bar{w}}$ inverse w.r.t. the circle $|w|=1$. In particular, the origin and the point at ∞ in the w-plane correspond to z and its conjugate with the changed sign.

Let the required transformation be $w=\frac{\alpha z+\beta}{\gamma z+\delta}$. Obviously $\gamma\neq 0$, otherwise the points at infinity in the two planes would correspond. Since $w=0$, $w=\infty$ correspond respectively to $z=-\frac{\beta}{\alpha}$, $z=-\frac{\delta}{\gamma}$, we can write $-\frac{\beta}{\alpha}=a$, $-\frac{\delta}{\gamma}=-\bar{a}$, and then $w=\frac{\alpha}{\gamma}\cdot\frac{z-a}{z+\bar{a}}$.

Now, the circle $|w|=1$ corresponds to the straight line $\mathrm{Re}(z)=0$; $\therefore$ the point $z=0$ must correspond to a point on the circle $|w|=1$, so that $\left|\frac{\alpha}{\gamma}\cdot\frac{-a}{\bar{a}}\right|=\left|\frac{\alpha}{\gamma}\right|=1$.

Thus, we may write $\alpha=\gamma e^{i\lambda}$, where λ is real, and obtain

$$w=\frac{z-a}{z+\bar{a}}e^{i\lambda}. \qquad \ldots\ldots(1)$$

Since $z=a$ gives $w=0$, a must be a point of the half-plane $\mathrm{Re}(z)>0$, *i.e.*, $\mathrm{Re}(a)>0$. With this condition, (1) is the required transformation.

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Note. It can be easily shown geometrically that (1) is the required transformation only if $\mathrm{Re}(a) < 0$;

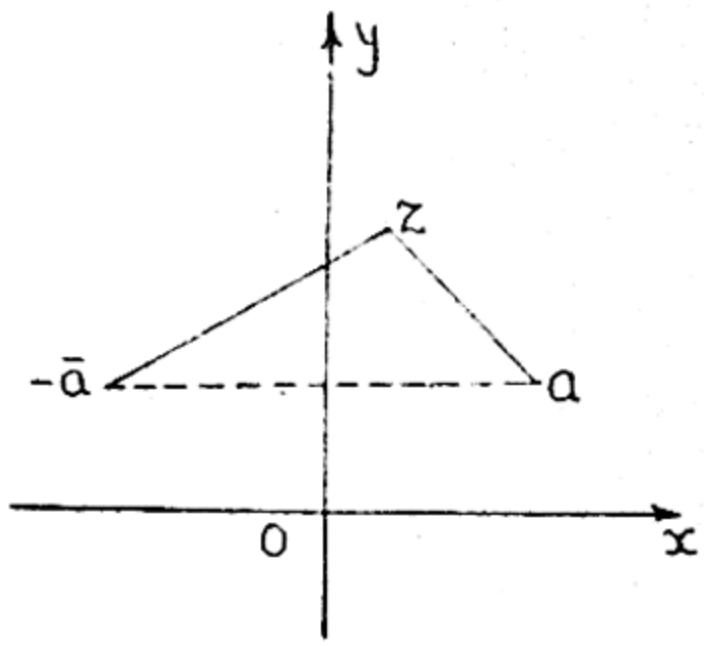


Fig. 15

for, $|w| < 1$ correspond to

$$\left|\frac{z-a}{z+\bar{a}}\right| < 1,$$

i.e., $|z-a| < |z+\bar{a}|$, which is possibly only if $\mathrm{Re}(a) > 0$.

If $\mathrm{Re}(a) \leqslant 0$, then the distance $|z-a| \geqslant |z+\bar{a}|$.

Ex. 3. *Find the bilinear transformation which transforms the circle* $|z| \leqslant \rho$ *into* $|w| \leqslant \rho'$.

Let the required transformation be $w = \dfrac{\alpha z+\beta}{\gamma z+\delta}$.

The points $w=0$, $w=\infty$ must correspond to the points $z=a$, $z=\rho^2/\bar{a}$, where $|a| < \rho$, since the interiors of the circles $|w| = \rho'$ and $|z| = \rho$ correspond. Hence, $-\beta/\alpha = a$, $-\delta/\gamma = \rho^2/\bar{a}$, and so

$$w = \frac{\alpha}{\gamma} \cdot \frac{z-a}{z-\rho^2/\bar{a}} = \frac{\alpha\bar{a}}{\gamma} \cdot \frac{z-a}{\bar{a}z-\rho^2}.$$

Further, since the circle $|w| = \rho'$ corresponds to $|z| = \rho$, the $z = \rho$ corresponds to a point on $|w| = \rho'$, so that

$$\rho' = |w| = \left|\frac{\alpha\bar{a}}{\gamma} \cdot \frac{\rho-a}{\bar{a}\rho-\rho^2}\right| = \frac{1}{\rho}\left|\frac{\alpha\bar{a}}{\gamma}\right|\left|\frac{\rho-a}{\bar{a}-\rho}\right| = \frac{1}{\rho}\left|\frac{\alpha\bar{a}}{\gamma}\right|.$$

Thus, we may write $\dfrac{\alpha\bar{a}}{\gamma} = \rho\rho' e^{i\lambda}$, where λ is real ;

hence, $w = \rho\rho' \dfrac{z-a}{\bar{a}z-\rho^2} e^{i\lambda}$ ($|a| < \rho$).(1)

This is the required transformation ; for, if $z = \rho e^{i\theta}$, $a = be^{i\beta}$ $[b = |a| < \rho]$,

then $|w| = \rho\rho' \left|\dfrac{\rho e^{i\theta} - be^{i\beta}}{\rho be^{i(\theta-\beta)} - \rho^2}\right|$

$$= \rho' \left|\frac{\rho e^{i\theta} - be^{i\beta}}{be^{i\theta} - \rho e^{i\beta}}\right| = \rho'.$$

If $z=re^{i\theta}$ where $r<\rho$, then.

$$\rho^2 \mid z-a \mid^2 - \mid \bar{a}z-\rho^2 \mid^2 = \rho^2 \{r^2-2rb\cos(\theta-\lambda)+b^2\} - \{b^2r^2-2br\rho^2\cos(\theta-\lambda)+\rho^4\}$$

$$=(r^2-\rho^2)(\rho^2-b^2)<0, \quad [\because r<\rho,\ \rho>b]$$

hence $\mid w \mid <\rho'$; *i.e.*, the interiors of the circles correspond.

Note 1. When $\rho=\rho'$, the tranformation $w=\rho^2\cdot\dfrac{z-a}{\bar{a}z-\rho^2}e^{i\lambda}$ ($\mid a \mid <\rho$) transforms the circle $\mid z \mid \leqslant\rho$ into the circle $\mid w \mid \leqslant\rho$. Since this is true for different sets of values of λ and a, it follows that the identical transformation $w=z$ is not the only transformation which can map a domain upon itself. The identical transformation $w=z$ is a particular case of (1) above and can be obtained by making $z=0$ correspond to $w=0$.

Note 2. When $\rho=\rho'=1$, the transformation

$$w=\frac{z-a}{\bar{a}z-1}e^{i\lambda}(\mid a \mid <1) \qquad \ldots\ldots(2)$$

transforms the unit circle $\mid z \mid \leqslant 1$ into the unit circle $\mid w \mid \leqslant 1$.
(*Agra 1935, '44, '52*)

Ex. 4. *Prove that, if* $w=\dfrac{\alpha z+\beta}{\gamma z+\delta}$ *and* $\alpha\delta-\beta\gamma=1$, *the linear magnification is* $\mid \gamma z+\delta \mid^{-2}$

Show that the circle $\mid \gamma z+\delta \mid =1$ $(\gamma\neq 0)$ *is the complete locus of points in the neighbourhood of which lengths and areas are unaltered by the transformation. Prove that the lengths and areas within this circle are increased and lengths and areas outside this circle are decreased in magnitude by the transformation.*

We have $w=\dfrac{\alpha z+\beta}{\gamma z+\delta}=\dfrac{\alpha}{\gamma}+\dfrac{\beta\gamma-\alpha\delta}{\gamma}\cdot\dfrac{1}{\gamma z+\delta}$

$$=\frac{\alpha}{\gamma}-\frac{1}{\gamma(\gamma z+\delta)}(\because \alpha\delta-\beta\gamma=1);$$

$\therefore \left\mid \dfrac{dw}{dz} \right\mid = \dfrac{1}{\mid \gamma z+\delta \mid^2}$, which gives the linear magnification.

Hence, the linear magnification is constant along the circle $\mid \gamma z+\delta \mid =1$ $(\gamma\neq 0)$, *i.e.*, every point on the locus $\mid \gamma z+\delta \mid =1$ is such that in its neighbourhood lengths and areas are unaltered by the transformation.

Again, as $\gamma w-\alpha=-\frac{1}{\gamma z+\delta}$, the region inside the circle $|\gamma z+\delta|=1$ is mapped on the outside of the circle $|\gamma w-\alpha|=1$ $(\gamma\neq 0)$ and the region outside the circle $|\gamma z+\delta|=1$ is mapped on the interior of the circle $|\gamma w-\alpha|=1$ $(\gamma\neq 0)$.

As $\left|\frac{dw}{dz}\right|=\frac{1}{|\gamma z+\delta|^2}$, it follows that for points within the circle $|\gamma z+\delta|=1$, $\left|\frac{dw}{dz}\right|>1$ and for points outside the circle $|\gamma z+\delta|=1$, $\left|\frac{dw}{dz}\right|<1$.

Hence, the transformation increases the lengths and areas within the circle $|\gamma z+\delta|=1$ and decreases the lengths and areas outside the circle $|\gamma z+\delta|=1$.*

***Note.** This circle is called the *isometric* circle.

3.5. Some Special Transformations. We shall now consider some important transformations which can be effected by elementary functions.

3.51. The Function $w=z^n$.

If $z=re^{i\theta}$ and $w=\rho e^{i\phi}$, then $\rho e^{i\phi}=rn\ e^{ni\theta}$. so that

$$\rho=r^n,\ \phi=n\theta. \qquad \ldots\ldots(1)$$

The distance from the origin is therefore raised to nth power, and the polar angle is multiplied by n. An angular region $\alpha<$ amp z $<\beta$ in the z-plane is mapped on the angular region $n\alpha<$ amp w $<n\beta$ in the w-plane. If $\beta-\alpha$ is greater than $\frac{2\pi}{n}$, the angular region in the w-plane covers part of this plane more than once. The ambiguity arising from this is removed, if the w-plane is cut along the real axis from $w=0$ to $w=\infty$.

From the equations (1) it follows that the circles $r=k$ of the z-plane and the circles $\rho=k^n$ of the w-plane correspond, and, in particular, the unit circle $|z|=1$ corresponds to the unit circle $|w|=1$.

3.52. The Function $w=z^2$. As in the previous article, the angular region $\alpha < \text{amp } z < \beta$ in the z-plane is mapped on the angular region $2\alpha < \text{amp } w < 2\beta$. If $\beta-\alpha > \pi$, the angular region in the w-plane covers part of this plane twice. To prevent this the w-plane is cut along the real axis from $w=0$ to $w=\infty$. Then the upper half or the lower half of the z-plane corresponds to the whole cut plane ; the positive real z-axis corresponds to the upper edge and the negative real z-axis to the lower edge of the cut in the w-plane.

If $w=u+iv$, $z=x+iy$, then $u+iv=(x+iy)^2$;

$$\therefore \quad u=x^2-y^2, \text{ and } v=2xy, \qquad \ldots\ldots(1)$$

Hence, the straight lines $u=$constant, $v=$constant in the w-plane correspond to the rectangular hyperbolas $x^2-y^2=$ constant, $2xy=$constant. The conformal property does not hold at $z=0$.

To the region $2xy > c_1$ in the v-plane corresponds the region $v > c_1$ in the w-plane; and to the region $2xy < c_2$ in the z-plane corresponds the region $v < c_2$.

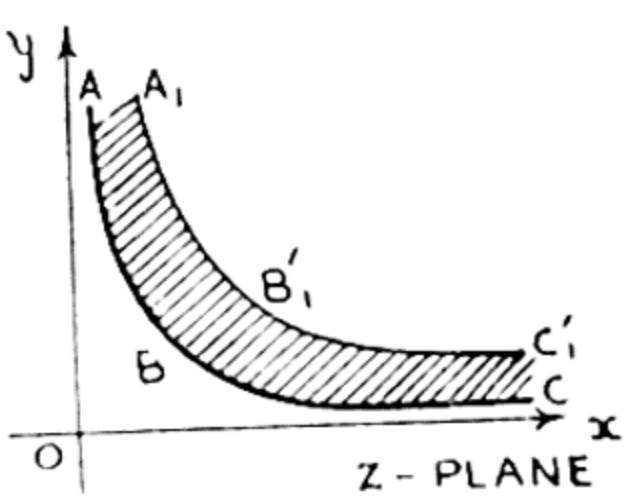


Fig. 16

Hence, $w=z^2$ maps the region between two hyperbolas in the z-plane on a parallel strip in the w-plane. In figures 16 and 17 shaded regions correspond.

As $B_1 \longrightarrow \infty$, $B'_1 \longrightarrow \infty$; and the interior of the hyperbola ABC is mapped on the half-plane above the line A′B′C′.

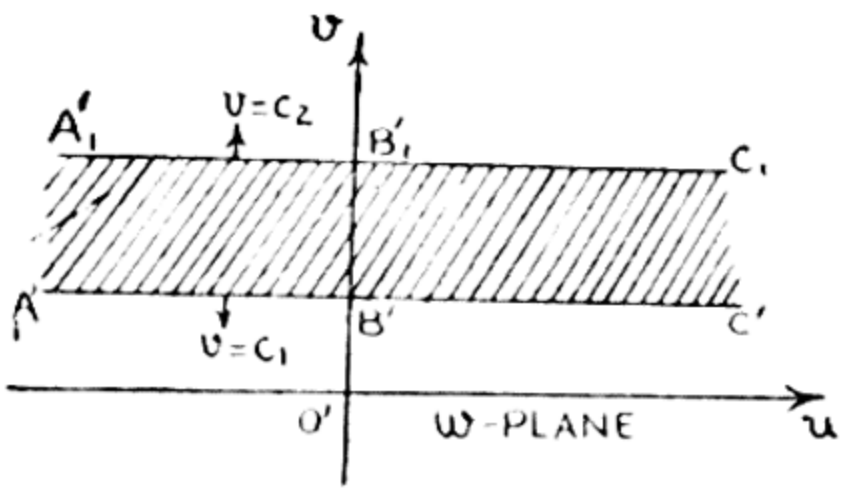


Fig. 17

Similarly, the region between the hyperbolas $x^2-y^2=c'$ and $x^2-y^2=c''$ in the z-plane is mapped on the infinite strip between the lines and $u=c'$ and $u=c''$ in the w-plane.

Hence, the rectangle formed by the lines $v=c_1$, $v=c_2$, $u=c'$, and $u=c''$ corresponds to the area enclosed by the four hyperbolas $2xy=c_1$, $2xy=c_2$, $x^2-y^2=c'$, and $x^2-y^2=c''$.

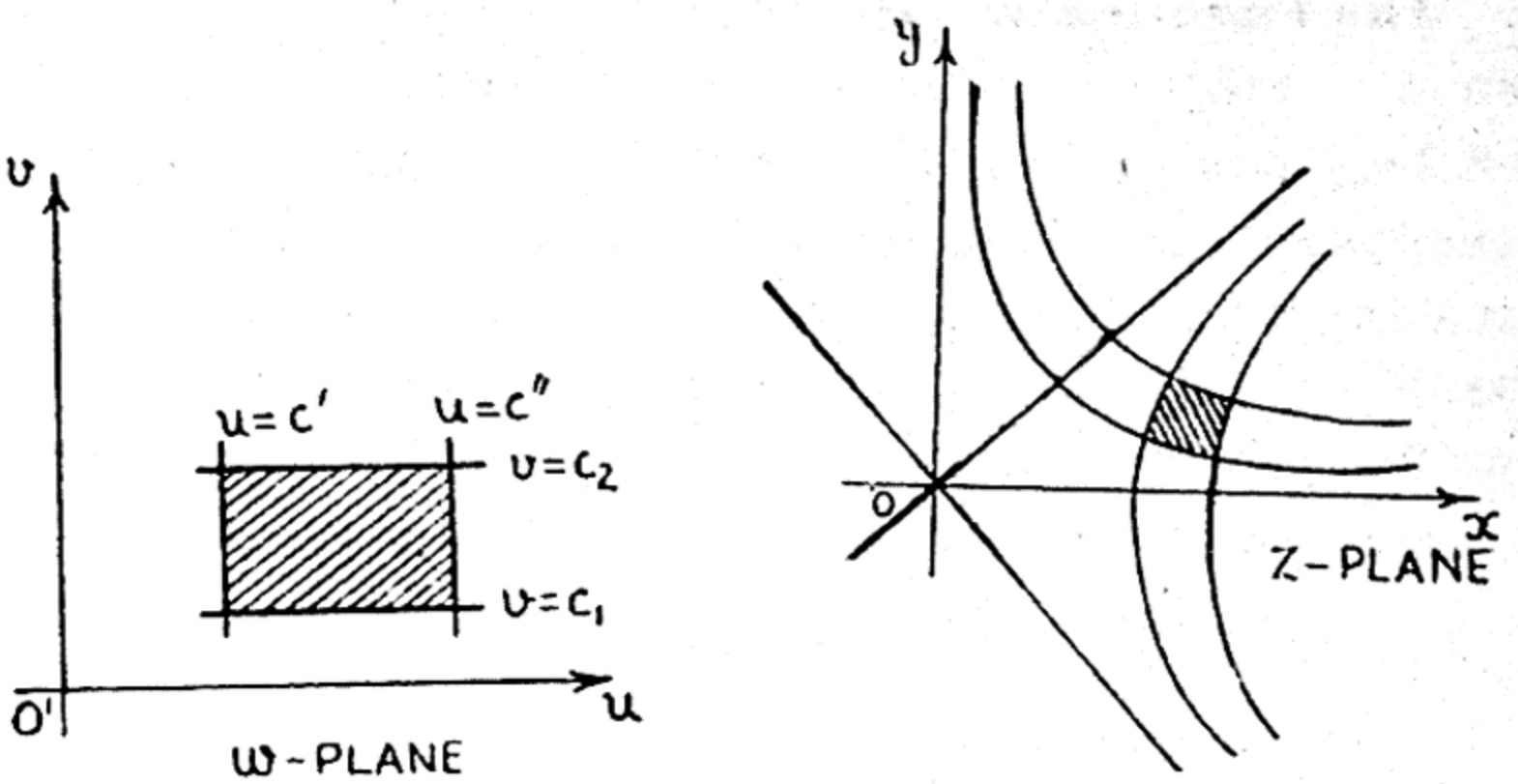


Fig. 18

It can be easily verified that the transformation $w=z^2$ maps the circles $|z-a|=k$, (a, k real) in the z-plane on the limacons $|w-a^2+k^2|=2ak+2k^2\cos\theta$ in the w-plane. When $a=k$, the limacon reduces to a cardioide.

Further, from equations (1), we have

$$v^2=4x^2(x^2-u),\ v^2=4y^2(y^2+u) \qquad \ldots\ldots(2)$$

Hence, the parabola $v^2=4c_1^2(c_1^2-u)$ corresponds to the straight lines $x=c_1$. The vertex of the parabola is at $u=c_1^2$ and the focus at the origin. To the orthogonal system of lines $y=c_2$, there corresponds another system of confocal parabolas with vertex at $u=-y^2$.

As a particular case, the line $x=1$ is mapped on the parabola

$$v^2=4(1-u). \qquad \ldots\ldots(3)$$

The infinite strip between the lines $x=1$ and $x=x_0>1$ in the z-plane is mapped on the region between the parabolas $v^2=4(1-u)$ and $v^2=4x_0^2(x_0^2-u)$ in the w-plane.

Making $x_0\longrightarrow\infty$, we see that the part of the z-plane to the right of the line $x=1$ is mapped on the exterior of the parabola (3).

If $x\longrightarrow 0$, the parabola $v^2=4x^2(x^2-u)$ narrows down to a slit along the negative real axis from $w=0$ to $w=-\infty$. Hence, *the half z-plane Re* $(z)\geqslant 0$ *corresponds to the whole* w-plane *cut along the real axis from* $w=0$ *to* $w=-\infty$. On the cut w-plane, the parabola (3) corresponds to the line $x=-1$ also and the exterior of the

parabola (3) corresponds to the portion of the z-plane to the left of the line $x=-1$.

Combination of $w=z^2$ with a bilinear transformation

$$z=\frac{2}{\zeta+1} \qquad \ldots\ldots(4)$$

gives us the transformation

$$w=\frac{4}{(\zeta+1)^2}, \qquad \ldots\ldots(5)$$

which maps the exterior of the parabola (3) on the interior of the unit circle in the ζ—plane. For, let $\zeta=\xi+i\eta$, then from (4) we have

$$z=x+iy=\frac{2(\xi+1-i\eta)}{(\xi+1)^2+\eta^2};$$

$\therefore$ the region $x>1$ is transformed into $\frac{2(\xi+1)}{(\xi+1)^2+\eta^2}>1$,

or $\xi^2+\eta^2<1$, *i.e.*, $|\zeta|<1$.

The transformation (5), however, cannot map the interior of the parabola (3) on the interior of the unit circle $|\zeta|=1$, as can be easily verified.

3.53. The Transformation $w=\tan^2\frac{z}{2}$.

The transformation can be written as $w=\frac{1-\cos z}{1+\cos z}$.

If $z=\frac{\pi}{2}+iy$, then $\cos z=-i\sinh y$, and $w=\frac{1+i\sinh y}{1-i\sinh y}$;

$\therefore$ $|w|=\frac{|1+i\sinh y|}{|1-i\sinh y|}=1$, and amp $w=2$ amp $(1+i\sinh y)$.

As y changes from $-\infty$ to ∞, sinh y changes from $-\infty$ to ∞, *i.e.*, amp $(1+i\sinh y)$ changes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Hence, amp w changes from $-\pi$ to π as y changes from $-\infty$ to ∞ on the line

$x=\frac{\pi}{2}$ in the z-plane ; so that w describes the whole unit circle once.

By writing $z=iy$, cos z=cosh y and w is real. Thus, as y

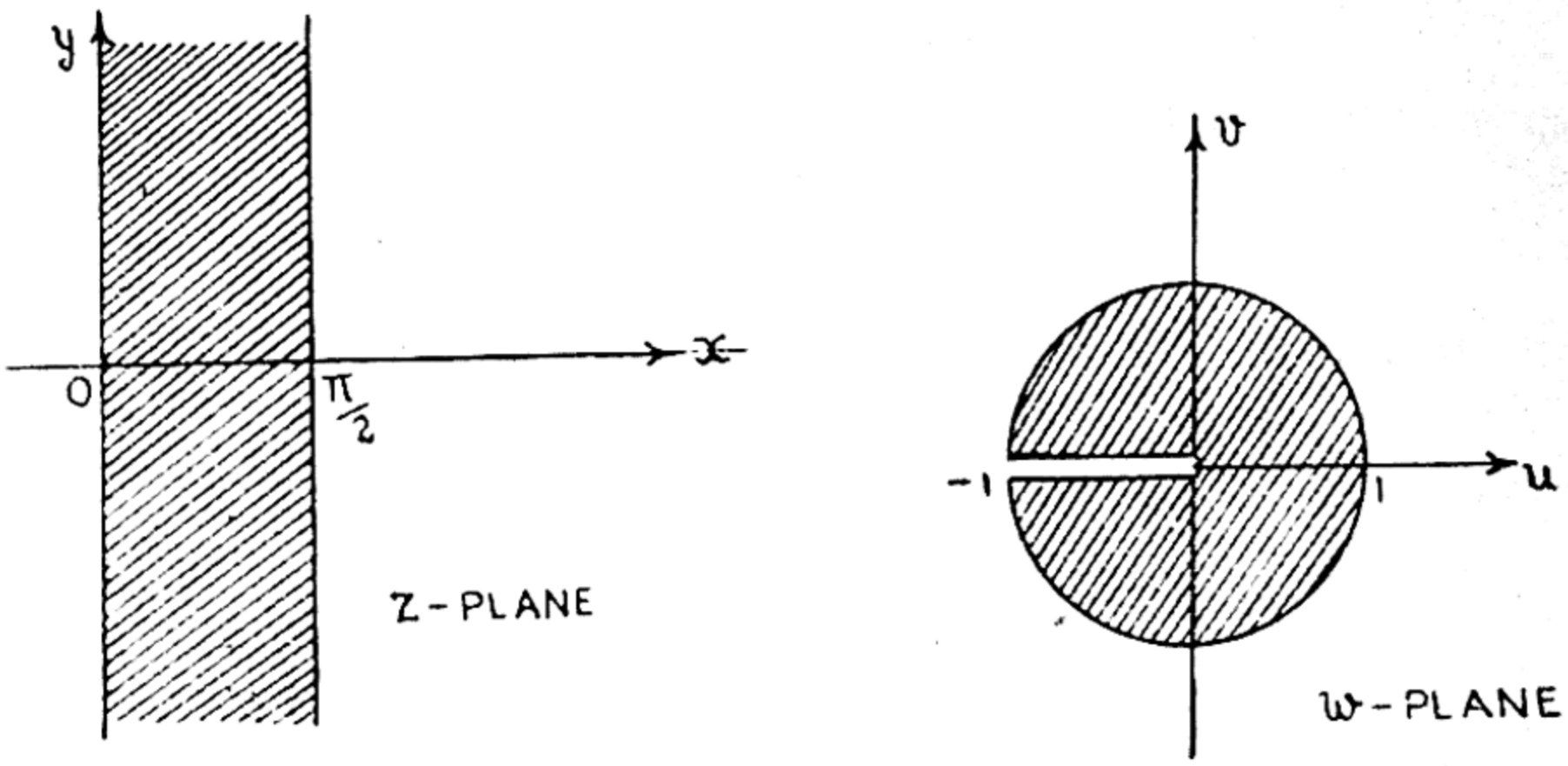


Fig. 19.

changes from ∞ to 0, w changes from -1 to 0, and as y changes from 0 to $-\infty$, w retraces its path from 0 to -1.

Hence, the strip between the lines $x=0$, $x=\frac{\pi}{2}$ *is mapped on the interior of the unit circle in the w-plane, cut along the real axis from* $w=-1$ *to* $w=0$, as illustrated in the figure. It is easy to show that the interior correspond.

For, $$w=\frac{1-\cos z}{1+\cos z}=\frac{1-\cos(x+iy)}{1+\cos(x+iy)}$$

$$=\frac{1-\cos x\cosh y+i\sin x\sinh y}{1+\cos x\cosh y-i\sin x\sinh y}.$$

When $0<x<\frac{\pi}{2}$, $0<\cos x<1$;

$$\therefore\quad 1-\cos x\cosh y<1+\cos x\cosh y.$$

Hence, $$|w|^2=\frac{(1-\cos x\cosh y)^2+\sin^2 x\sinh^2 y}{(1+\cos x\cosh y)^2+\sin^2 x\sinh^2 y}<1.$$

3.54. The Transformation $w=\tan^2(\frac{1}{4}\pi\sqrt{z})$.

We have already seen that the transformation $w=z^2$ (§ 3.52) maps the strip between the lines $x=0$ and $x=1$ on the interior of the parabola $v^2=4(1-u)$ in the w-plane cut along the real axis from $w=0$ to $w=-\infty$, *i.e.*, the transformation $z=w^2$, or $w=\sqrt{z}$ maps the strip between the lines $u=0$ and $u=1$ in the w-plane on the interior of the parabola $y^2=4(1-x)$ in the z-plane cut along the real axis from $z=0$ to $z=-\infty$.

Hence, combining the transformation of § 3.53, $w=\tan^2(\frac{1}{2}\zeta)$ with $\zeta=\frac{1}{2}\pi t$, $t=\sqrt{z}$, where $\zeta=\xi+i\eta$, $t=\sigma+it$, we easily see that the interior of the unit circle in the cut w-plane corresponds to the interior of the parabola $y^2=4(1-x)$ in the cut z-plane. The cuts in the w-plane and the z-plane correspond. The cuts in the w-plane and the z-plane are not needed for direct transformation ; they are needed only for the intermediate transformation.

Since $\frac{dw}{dz}\neq 0$ as $z\to 0$, the mapping is conformal and one-to-one.

3.6. Exponential and Logarithmic Transformations. The function $w=e^z$.....(1) gives rise to two important special transformations. Writing $z=x+iy$ and $w=\rho e^{i\phi}$, we have $\rho=e^x$, $\phi=y$. A horizontal strip of the z-plane bounded by the lines $y=y_1$ and $y=y_2$, where $|y_1-y_2|<2\pi$, is transformed into a wedge-shaped region of the w-plane, the angle of the wedge being $\alpha=|\phi_2-\phi_1|=|y_2-y_1|$. The mapping is conformal throughout the interior of these regions, since the derivative of e^z is never zero.

As a special case, if $|y_2-y_1|=\pi$ (*e.g.* $y_1=0$, $y_2=\pi$), the wedge becomes a half-plane.

The restriction on the width of the strip, namely that $|y_1-y_2|<2\pi$, may be dropped. If, for example, $y_1=0$ and $y_2=2\pi$, the strip is represented on the w-plane cut along the positive real axis ; and if $|y_1-y_2|>2\pi$, the wedge obtained covers part of the w-plane more than once and the cut-plane is to be used.

We obtain the second special transformation furnished by (1) by considering an arbitrary vertical strip bounded by the lines $x=x_1$, $x=x_2$. This strip is mapped in the cut w-plane on the

annular region $\rho_1 < |w| < \rho_2$ an infinite number of times. If now, keeping x_2 constant, we let $x_1 \to -\infty$, the strip $x_1 < x < x_2$ becomes in the limit the half-plane $\mathrm{Re}(z) < x_2$, and instead of the annular region in the cut w-plane we obtain the circle $|w| < \rho_2$ with a logarithmic branch point at $w=0$.

The function inverse to (1), namely $w = \log z$(2) gives, of course, on interchanging the z-plane and w-plane, precisely the same conformal mapping as (1).

Combining both the special transformations discussed above we see that the rectangular region $y_1 \leqslant y \leqslant y_2$, $x_1 \leqslant x \leqslant x_2$ is mapped on the region $y_1 \leqslant \phi \leqslant y_2$, $e^{x_1} \leqslant \rho \geqslant e^{x_2}$, bounded by circles and rays.

In particular, if $y_1=0$, $y_2=\pi$, the rectangle $0 \leqslant y \leqslant \pi$ maps on half of a circular ring.

The semi-infinite strip $-\infty \leqslant x \leqslant 0$, $0 \leqslant y \leqslant \pi$ maps on the semi-circle $\rho \leqslant 1$, $0 \leqslant \phi \leqslant \pi$, *i.e.*, the unit semi-circle in the upper half of the w-plane.

3.7. The Transformation $\mathbf{w = \frac{1}{2}(z + \frac{1}{z})}$.

When $z=0$, w becomes infinite, and since $\frac{dw}{dz} = \frac{1}{2}(1 - \frac{1}{z^2})$, the points $z = \pm 1$, at which $\frac{dw}{dz} = 0$, are the critical points of the transformation.

Putting $z = re^{i\theta}$, $w = u + iv$, we have $2u = (r + \frac{1}{r}) \cos\theta$,

$2v = (r - \frac{1}{r}) \sin\theta$, and on eliminating θ, we get the ellipse in the

w-plane $$\frac{u^2}{\frac{1}{4}\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r + \frac{1}{r}\right)^2} = 1. \qquad \ldots\ldots(1)$$

corresponding to each of the two circles $|w| = r$, $|w| = \frac{1}{r}$. As $r \to 1$, the major semi-axis of the ellipse $\to 1$, while the minor semi-

axis $\to\infty$. As $r\to 0$, or as $r\to\infty$, both semi-axes $\to\infty$. It follows that the inside and the outside of the unit circle in the z-plane both correspond to the whole w-plane cut along the real axis from -1 to 1. The unit circle $|z|=1$ itself corresponds to a very narrow ellipse in the w-plane, which is the cut along the real axis enclosing the critical points -1 and 1.

3.71 The Transformation $\mathbf{w=\frac{1}{2}\left\{(a-b)z+\frac{a+b}{z}\right\}}$.

Putting $z=re^{i\theta}$, and $w=u+iv$, we have

$$u=\tfrac{1}{2}\left\{(a-b)\,r+\frac{a+b}{r}\right\}\cos\theta,\quad v=\tfrac{1}{2}\left\{(a-b)r-\frac{a+b}{r}\right\}\sin\theta, \qquad \ldots\ldots(1)$$

and eliminating θ,

$$\frac{u^2}{\frac{1}{4}\left\{(a-b)r+\frac{a+b}{r}\right\}^2}+\frac{v^2}{\frac{1}{4}\left\{(a-b)r-\frac{a+b}{r}\right\}^2}=1. \qquad \ldots\ldots(2)$$

The equation (2) represents a system of confocal ellipses corresponding to the concentric circles $|z|=r$, the distance between the foci being $2\sqrt{a^2-b^2}$.

Eliminating r from equations (1), we have

$$u^2\sec^2\theta-v^2\operatorname{cosec}^2\theta=a^2-b^2,\ \text{or}\ \frac{u^2}{(a^2-b^2)\cos^2\theta}-\frac{v^2}{(a^2-b^2)\sin^2\theta}=1.$$

Hence, straight lines through the origin in the z-plane correspond to confocal hyperbolas, the distance between the foci being $2\sqrt{a^2-b^2}$.

3.8. The Transformation $\mathbf{w=\sin^{-1}z}$.

We have $z=\sin w.$ $\qquad\ldots\ldots(1)$

Putting $z=x+iy$, $w=u+iv$, we get

$$x=\sin u\cosh v, y=\cos u\sinh v. \qquad \ldots\ldots(2)$$

From equations (2) we infer that $\dfrac{x^2}{\cosh^2 v}+\dfrac{y^2}{\sin^2 v}=1,$ $\ldots\ldots 3)$

$$\text{and}\ \frac{x^2}{\sin^2 u}-\frac{y^2}{\cos^2 u}=1. \qquad \ldots\ldots(4)$$

Thus the st. lines v=constant go over into confocal ellipses with their foci at the points $x=\pm 1$, $y=0$; and the straight lines $u=$ constant go over into confocal hyperbolas with the same foci.

In particular, consider the strip in the w-plane bounded by the lines $u=\pm\frac{\pi}{2}$; with $v\geqslant 0$. A horizontal line segment $-\frac{\pi}{2}\leqslant u\leqslant \frac{\pi}{2}$, $v=v_1>0$ maps on the semi-ellipse $\frac{x^2}{\cosh^2 v_1}+\frac{y^2}{\sinh^2 v_1}=1$, which lies in the upper half-plane : $x=\cosh v_1 \sin u$, $y=\sinh v_1 \cos u$.

Similarly, a vertical line segment $u=u_1$, $\frac{-\pi}{2}<u_1<\frac{\pi}{2}$; $v>0$ maps on a half-branch of the hyperbola $\frac{x^2}{\sin^2 u_1}-\frac{y^2}{\cos^2 u}=1$, where $x=\sin u_1 \cosh v$, $y=\cos u_1 \sinh v$.

Each point in the upper half of the z-plane is the intersection of just one pair of curves of this orthogonal system of ellipses and hyperbolas and corresponds to just one point in the semi-infinite strip $\frac{-\pi}{2}\leqslant u\leqslant\frac{\pi}{2}$, $v\geqslant 0$ in the w-plane. Also to each point of the strip there corresponds just one point z. Hence, the mapping of the strip $\frac{-\pi}{2}\leqslant u\leqslant\frac{\pi}{2}$, $v\geqslant 0$ upon the upper half of the z-plane is one-to-one. The right-hand half of the strip maps on the first quadrant of the z-plane and the left-hand half of this strip maps on the second quadrant of the z-plane.

The rectangle $-\pi\leqslant u\leqslant\pi$, $v_1\leqslant v\leqslant v_2$ maps on the region bounded by two confocal ellipses and if $v_1>0$, the elliptic ring has a cut along the negative real axis.

Ex. 1. *Discuss the application of the transformation* $w=z^2$ *to the area in the first quadrant of the* z*-plane bounded by the axes and the circles* $|z|=a$, $|z|=b$ $(a>b>0)$. *Is the transformation conformal ?*

(Agra 1947)

Putting $w=u+iv$ and $z=x+iy$, we have $u=x^2-y^2$, $v=2xy$;

$\therefore \quad u^2+v^2=(x^2+y^2)^2$, or $|w|=|z|^2$.

Hence, the circles $|z|=a$, $|z|=b$ map on the circles $|w|=a^2$, $|w|=b^2$ respectively.

If $z=re^{i0}$ and $w=\rho e^{i\phi}$, then $\rho=r^2$ and $\phi=2\theta$. Hence, the region $0\leqslant\theta\leqslant\frac{\pi}{2}$ maps on the region $0\leqslant\phi\leqslant\pi$ and the area in the first quadrant of the z-plane bounded by the axes and the circles $|z|=a$, $|z|=b$ on the area in the upper half of the w-plane bounded by the real axis and the circles $|w|=a^2$, $|w|=b^2$.

The transformation discussed is conformal, since $w=z^2$ is analytic and $\frac{dw}{dz}\neq 0$ in the given area.

Ex. 2. *Discuss the mapping upon the w-plane of the line $y=mx$ in the z-plane by means of the relation $w=e^z$, m being not equal to zero.* *(Agra, 1933)*

Putting $w=\rho e^{i\phi}$, and $z=x+iy$, we have $\rho e^{i\phi}=e^{x+iy}$;

$\therefore \quad \rho=e^x$, and $\phi=y$.

Hence, the straight line $y=mx$ in the z-plane is mapped on the curve $\phi=m\log\rho$, or $\rho=e^{\phi/m}$, which is a spiral, since $m\neq 0$.

Ex. 3. *If $w=\log\frac{z-1}{z+1}$, find the curves on the z-plane corresponding to $u=0$ and $v=0$ on the w-plane, where $w=u+iv$.* *(Agra, 1936)*

The given transformation is a combination of

$$w=\log\zeta, \quad \zeta=\frac{z-1}{z+1}.$$

Let $\zeta=\xi+i\eta$, then $\xi+i\eta=e^{u+iv}$, *i. e.*, $\xi=e^u\cos v$, $\eta=e^u\sin v$.(1)

Eliminating v, we have $\xi^2+\eta^2=e^{2u}$; hence $u=0$ corresponds to $\xi^2+\eta^2=1$, *i. e.*, $|\zeta|=1$. Further, the transformation $\zeta=\frac{z-1}{z+1}$ maps $|\zeta|=1$ on $\left|\frac{z-1}{z+1}\right|=1$, which represents the imaginary

axis $x=0$ in the z-plane. Thus, the line $u=0$ in the w-plane corresponds to the line $x=0$ in the z-plane.

From (1) we see that $v=0$ corresponds to $\eta=0$. Now,

$$\xi+i\eta=\frac{z-1}{z+1}=1-\frac{2(x+1-iy)}{(x+1)^2+y^2}\ ;$$

$\therefore$ $\eta=0$ corresponds to $y=0$. Hence, the line $v=0$ in the w-plane corresponds to the line $y=0$ in the z-plane.

Aliter. We have $u=\log\left|\frac{z-1}{z+1}\right|$, *i.e.*, $\left|\frac{z-1}{z+1}\right|=e^u$;

$\therefore$ $u=0$ corresponds to $\left|\frac{z-1}{z+1}\right|=1$, *i.e.*, $x=0$.

Also $v=\text{amp}\,(z-1)-\text{amp}\,(z+1)=\tan^{-1}\frac{y}{x-1}-\tan^{-1}\frac{y}{x+1}$

$$=\tan^{-1}\frac{2y}{x^2+y^2-1}\ ;\quad \therefore\ v=0 \text{ corresponds to } y=0.$$

Ex. 4. *If $w=2z+z^2$, prove that the circle $|z|=1$ corresponds to a cardioide in the w-plane, and that an equilateral triangle in the circle corresponds to the points of contact of parallel tangents of the cardioide.* *(Agra 1941,'53)*

The given transformation is a combination of

$$w=\zeta-1, \text{ and } \zeta=(z+1)^2.$$

Let $z=e^{i0}$ and $\zeta=\rho e^{i\phi}$. so that $|z|=1$,

and $\rho e^{i\phi}=(e^{i0}+1)^2=e^{i0}.(e^{i\theta/2}+e^{-i\theta/2})^2=4\cos^2\frac{\theta}{2}e^{i0}$

$$=2\,(1+\cos 0)\,e^{i0}\ ;$$

$\therefore$ $\phi=0$, and $\rho=2\,(1+\cos 0)=2\,(1+\cos\phi)$, which is the equation of a cardioide in the ζ-plane.

The transformation $w=\zeta-1$ leaves the configuration unchanged on mapping from the ζ-plane to the w-plane. Hence, the given transformation maps the circle $|z|=1$ on a cardioide in the w-plane.

The vertices of any equilateral triangle in $|z|=1$ have

vectorial angles equal to $0.\ 0+\frac{2\pi}{3},\ 0+\frac{4\pi}{3}$ respectively. The corresponding points on the cardioide are $[2\ (1+\cos 0),\ 0]$,

$$\left[2\ (1+\cos\overline{0+\tfrac{2\pi}{3}}),\ 0+\frac{2\pi}{3}\right] \text{ and } \left[2\ (1+\cos\overline{0+\tfrac{4\pi}{3}}),\ 0+\frac{4\pi}{3}\right].$$

$$\text{Now, } \frac{dy}{dx} \text{ at } [2a\ (1+\cos 0),\ 0] = \frac{\frac{d}{d0}\ [2a\ (1+\cos 0)\sin 0]}{\frac{d}{d0}\ [2a\ (1+\cos 0)\cos 0]}$$

$$= \frac{\cos 0\ (1+\cos 0\)-\sin^2 0}{-\sin 0.\ (1+\cos 0)-\sin 0\cos 0} = -\frac{\cos 0+\cos 20}{\sin 0+\sin 20} = -\cot\frac{30}{2}.$$

$$\text{Similarly, } \frac{dy}{dx} \text{ at } \left[(2a\ (1+\cos\overline{0+\tfrac{2\pi}{3}}),\ 0+\frac{2\pi}{3}\right]$$

$$= -\cot\frac{3\left(0+\frac{2\pi}{3}\right)}{2} = -\cot\frac{3\,0}{2},$$

$$\text{and } \frac{dy}{dx} \text{ at } \left[2a\ (1+\cos\overline{0+\tfrac{4\pi}{3}}),\ 0+\frac{2\pi}{3}\right] = -\cot\frac{3\left(0+\frac{2\pi}{3}\right)}{2}$$

$$= -\cot\frac{3\,0}{2}.$$

Hence, the three tangents are parallel. Thus, an equilateral triangle in the circle $|z|=1$ corresponds to points of contact of parallel tangents of the cardioide.

Ex. 5. *Show that the transformation* $\frac{z}{c}=\left(\frac{1+w}{1-w}\right)^{2a/\pi i}$ *transforms the inside of the circle* $|w|=1$ *with two semi-circular indents, of centres 1 and* -1*, drawn so as to exclude these points from the circular area and boundary, into the annulus between two circles in the z-plane, of centre the origin and radii* ce^{a}, ce^{-a}, *with a single slit along the real axis.*

The given transformation is a combination of

$$\frac{z}{c}=\zeta^{2a/\pi i}, \ldots(1) \text{ and } \zeta=\frac{1+w}{1-w} \ldots(2)$$

From (2), we have $w=\dfrac{\zeta-1}{\zeta+1}$;

$\therefore \mid w \mid \leqslant 1$ is mapped on the half-plane

$\left| \dfrac{\zeta-1}{\zeta+1} \right| \leqslant 1$, *i.e.* $\xi \geqslant 0$, where $\zeta=\xi+i\eta$.

The semi-circular indent at 1, *i.e.*, $\mid w-1 \mid =\varepsilon_1$, where ε_1 is an arbitrarily small positive number, corresponds to

$\left| \dfrac{2}{\zeta+1} \right| =\varepsilon_1$ or $\mid \zeta+1 \mid =\dfrac{2}{\varepsilon_1}=R$, where R is a large $+ve$ number. Thus, the semi-circular indent at 1 corresponds to a very large circle, centre -1, in the ζ-plane.

Similarly, the semi-circular indent at -1 corresponds to a very small circle, centre $\zeta=0$, in the ζ-plane.

Hence, the inside of the circle $\mid w \mid =1$ indented at $w=1$, $w=-1$ so as to exclude the points and the semi-circular boundary of indents is mapped on the region $\mid \zeta \mid =\varepsilon$, $\mid \zeta+1 \mid =R$, $\xi \geqslant 0$ in the ζ-plane, where ε, is an arbitraily small positive number and R is an arbitrarily large positive number, *i.e.*, $\varepsilon \longrightarrow 0$ and $R \longrightarrow \infty$.

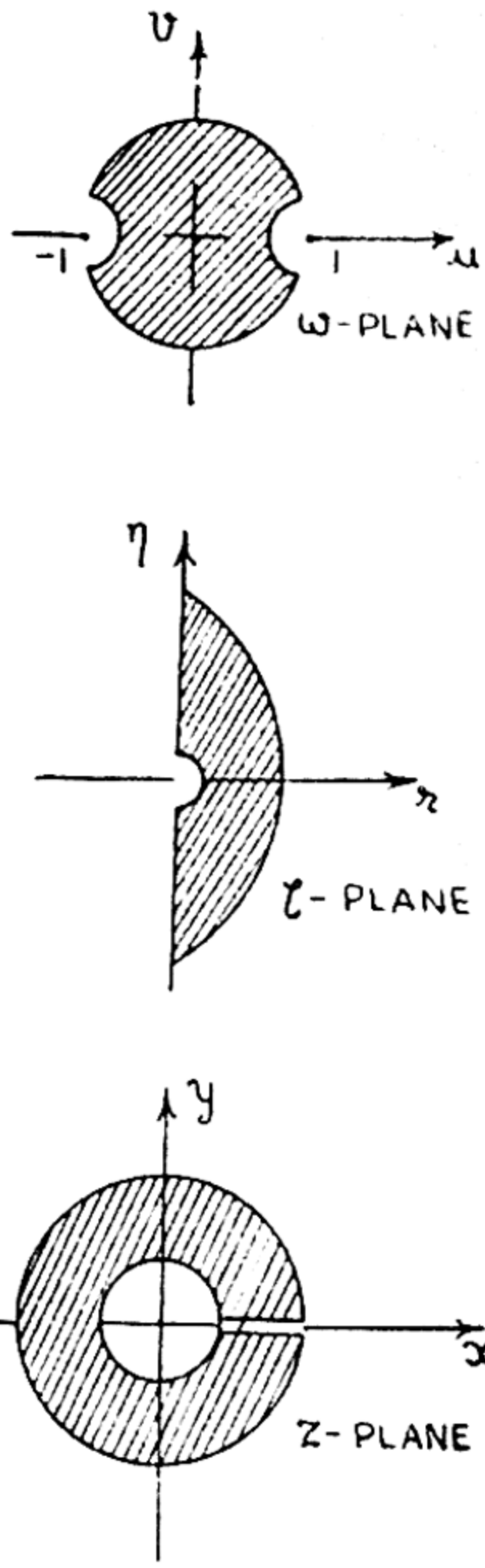


Fig. 20

From (1), we have $\dfrac{r}{c}e^{i\theta}=\left(\rho e^{i\phi}\right)^{2a/\pi i}$, where $z=re^{i\theta}$,

and $\zeta=\rho e^{i\phi}$, *i. e.*, $r=ce^{2a\phi/\pi}$, and $\theta=\dfrac{-2a}{\pi}\log \rho$.

$$\left[\because \rho^{2a/\pi i} = e^{\frac{2a}{\pi i}\log \rho} = e^{-i\frac{2a}{\pi}\log \rho}\right]$$

Hence, $\xi \geqslant 0$, *i. e.*, $-\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2}$ corresponds to the annulus between the circle $r=ce^{-a}$ and $r=ce^{a}$. When $\rho \to 0$, $0 \to \infty$ and when $\rho \to \infty$, $0 \to -\infty$, supposing a to be $+ve$.

Hence, the corresponding region in the z-plane is the annulus between the circles $|z| = ce^{a}$, $|z| = ce^{-a}$ with a cut along the positive real axis, or negative real axis.

Exercise 3

1. Show that the transformation $w=\frac{2z+3}{z-4}$ changes the circle $x^2+y^2-4x=0$ into the line $4u+3=0$, and explain why the curve obtained is not a circle.

2. Find the bilinear transformation which makes the points $1, i, -1$ in the z-plane correspond respectively to the points $0, 1, \infty$ in the w-plane.

3. In the transformation $z=\frac{i-w}{i+w}$ show that the upper half w-plane corresponds to the unit circle $|z| \leqslant 1$ in the z-plane.

[**Hint.** The upper half w-plane is given by $|w-i| \leqslant |w+i|$.]

4. Find the bilinear transformation which makes the points $2, 1+i, 0$ in the z-plane correspond respectively to the points $0, 1, \infty$ in the w-plane.

Show by sketches the domains of the w-plane and z-plane which correspond.

5. Find the bilinear transformation which maps the circle $|z| \leqslant 1$ on $|w-1| \leqslant 1$ and makes the points $z=0, 1$ correspond to $w=\frac{1}{2}, 0$ respectively.

Is the transformation uniquely determined by the data ?

6. Prove that $w=\frac{1+iz}{i+z}$ maps the part of the real axis between $z=1$ and $z=-1$ on a semi-circle in the w-plane.

Find all the figures that can be obtained from the originally selected part of the x-axis by successive applications of this transformation.

Also apply the transformation to the areas in the z-plane which are respectively inside and outside the unit circle $|z| = 1$.

(*Agra 1948*)

7. Express the relation $w=\dfrac{13\,i\,z+75}{3\,z-5\,i}$ in the form

$$\frac{w-a}{w-b}=k\,\frac{z-a}{z-b},$$ where a, b, k are constants.

Show that the circle $|z|=5$ is transformed into the circle in the w-plane on the line joining the points $w=a$ and $w=b$ as diameter and that points in the z-plane exterior to $|z|=5$ are transformed into points in the w-plane within the latter circle.
(*Agra 1949*)

8. Show that the equations $\left|\dfrac{z-a}{z-b}\right|=k\ (k\neq 1)$,

$\operatorname{amp}\dfrac{z-b}{z-a}=0$, where k and 0 are variable parameters, represent two orthogonal families.

9. Find the bilinear transformation which maps the half-plane $\operatorname{Im}(z)\geqslant 0$ on the unit circle $|w|\leqslant 1$.

10. Show that the transfomation $w=\dfrac{r(z-a)}{r^2-\bar{a}z}$ maps the circle $|z|=r$ on the unit circle $|w|=1$, if $|a|\neq r$. If $|a|<r$, it maps $|z|<r$ on $|w|<1$ and $|z|>r$ on $|w|>1$. If $|a|>r$, it maps $|z|>r$ on $|w|<1$ and $|z|<r$ on $|w|>1$.

11. Find what regions of the w-plane correspond by the transformation $w=\dfrac{z-i}{z+i}$ to : (i) the interior of a circle of centre $z=-i$, (ii) the region $y>0$, $x>0$, $|z+i|<2$. Illustrate by diagrams.

Show that the magnification is constant along any circle with $z=-i$ as centre.

12. Show that the transformation $w=k\dfrac{z-b}{z-a}$ maps the space bounded by three circular arcs in the z-plane on a rectilinear triangle in the w-plane.

13. Let C_1, $|z-z_1|=r_1$ and C_2, $|z-z_2|=r_2$ be two non-concentric circles in the z-plane, C_1 lying entirely within C_2. Show that, if $z=a$, $z=b$ are the limiting points of the system of coaxal

circles determined by C_1 and C_2, then $w=\dfrac{k\ (z-b)}{z-a}$ transforms

C_1 and C_2 into concentric circles in the w-plane with centres at $w=0$. If the radii of these concentric circles are ρ_1 and ρ_2, show that, although there is an infinite number of such representations, $\rho_1 : \rho_2$ is constant.

14. Find the bilinear transformation which maps the outside of the circle $|z|=1$ on the half-plane Re $(w) \geqslant 0$, so that the points $z=1,-i,-1$ correspond to $w=i,\ 0,-i$ respectively. What corresponds in the w-plane to : (i) the lines amp $z=$constant, $|z| \geqslant 1$, (ii) the concentric circles $|z|=r$, $(r>1)$?

15. Show that the transformation $w\,(z+i)^2=1$ maps the interior of the circle $|z|=1$ in the z-plane on the domain exterior to the circle $|z|=1$ in the z-plane on the domain exterior to the parabola $\dfrac{1}{2\rho}=1-\cos\phi$, where $w=\rho e^{i\phi}$, in the w-plane.

(Agra 1942)

16. If $w=\left(\dfrac{z-c}{z+c}\right)^2$, where c is real and positive, find the areas of the z-plane of which the upper half of the w-plane is the conformal representation. *(Agra 1934, '39)*

17. Show that by means of the transformation $w=\left(\dfrac{z-ic}{z+ic}\right)^2$,

(c real), the upper half of the w-plane can be made to correspond to the interior of a semi-circle in the z-plane.

(Agra 1951, '57)

18 Prove that, if $w=x+i\dfrac{b}{a}y$, $0<a<b$, the inside of the circle $x^2+y^2=a^2$ corresponds to the inside of an ellipse in the w-plane but that the transformation is not conformed.

19. Prove that for the transformation $w^2=(z-\alpha)\ (z-\beta)$, the critical points are $z=\alpha$, $z=\beta$, $z=\frac{1}{2}\ (\alpha+\beta)$, $w=0$, $w=\pm\frac{1}{2}\ i\ (\alpha-\beta)$.

Show also that the condition that $a=\infty$ is not a critical point of the transformation $w=f(z)$ is that $\lim\limits_{z\to\infty} z^2 f'(z)$ must be finite and non-zero.

20. Show that the length of the curve into which the semi-circular

arc $|z|=1, \frac{-\pi}{2} \leqslant \text{amp } z \leqslant \frac{\pi}{2}$ is transformed by

$w=\frac{4}{(z+1)^2}$ is $2\sqrt{2}+2\log(1+\sqrt{2})$.

21. Show that by the transformation $w=\left(\frac{z+i}{z-i}+1\right)^2$, the real axis in the z-plane corresponds to a cardioide in the w-plane. Indicate the region of the z-plane which corresponds to the interior of the cardioide.

22. If $w=-ic\cot\frac{z}{2}$, where c is real, show that the rectangle bounded by $x=0$, $x=\pi$, $y=0$, $y=\infty$, is conformally mapped on a quarter of the w-plane.

Find a transformation $\zeta=f(z)$ which maps this infinite rectangle on the semi-circle $|\zeta| \leqslant a$, $\xi \geqslant 0$ $(\zeta=\xi+i\eta)$.

[**Hint.** For the second part combine $w=-ic\cot\frac{z}{2}$ and

$w=\frac{\zeta-a}{\zeta+ia}$.]

23. By the relation $w=ae^{iz}$, where a is real, show that the rectangular strip on the z-plane lying between $y=\infty$, $y=-\infty$, $x=0$ and $x=\pi$, is conformally represented on the positive half of the w-plane. *(Agra 1937)*

24. If a is real and $0<b<\pi$, find the area of the domains in the w-plane which correspond by the transformation $w=e^z$ to the rectangle $a-b \leqslant x \leqslant a+b$, $-b \leqslant y \leqslant b$. Find the ratio of the areas of the two corresponding domains and prove that the ratio $\to e^{2a}$ as $b \to 0$.

25. Show that the transformation $w=e^{\pi z/a}$ maps an equiangular spiral in the w-plane on a straight line in the z-plane.

26. Show that $2w=\log\frac{1+z}{1-z}$ maps $|z|<1$ on the strip of the w-plane $-\frac{\pi}{4}<v<\frac{\pi}{4}$.

27. Show that the $iw=\log\left(\sqrt{\frac{z}{a}}-1\right)$ represents the strip $v=0$, $v=\infty$, $u=-\pi$, $u=\pi$ on the interior of the cardioide $r=2a\,(1+\cos\theta)$ in the z-plane cut along the real axis from the cusp to $x=a$.

28. Show that, if c is real, $\frac{z-c}{z+c}=ie^{iw}$ conformally transforms the strip $v=-\infty$, $v=\infty$, $u=0$, $u=\pi$ into the circle $|z|\leqslant c$.

29. Show that the transformation $w^2=1+e^z$ maps the lines $x=$constant on a series of confocal lemniscates (Cassini's ovals) in the w-plane.

30. If $w=\tanh z$, show that the lines $x=$const., $y=$const., correspond to coaxal circles in the w-plane.

Prove that this transformation maps the strip $0<y<\frac{\pi}{2}$ conmally on the upper-half of the w-plane.

31. Prove that $w=\cos z$ maps the strip $0<\text{Re}\,(z)<\pi$ conformally on the whole w-plane, supposed cut along the real axis from $-\infty$ to -1 and from $+1$ to $+\infty$.

32. If $z=x+iy$, prove that the inside of the parabola $y^2=4c^2(x+c^2)$ is mapped on the upper half of the w-plane by

$$w=i\,\cosh\frac{\pi\sqrt{z}}{2c}.$$

33. Prove that $z=c\,\cos\,(\beta\,\log w)$, where $c^2=a^2-b^2$, $\cosh\frac{\beta\pi}{2}=\frac{a}{c}$ gives the conformal representation of the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y}{b^2}=1$ and the two lines joining the foci to the ends of the major axis on the region in the w-plane defined by $u>0$, $1<|w|<e^{\pi/\beta}$. *(Agra 1931)*

34. Show that the transformation $w=z+\frac{1}{z}$ maps :

(i) the interior of the semi-circle $|z|=1$ in the upper half of the z-plane on the lower-half w-plane and the exterior of the

semi-circle $|z|=1$ in the upper-half z-plane on the upper-half w-plane ;

(ii) the semi-circular ring between the circles $|z|=1$, $|z|=k>1$ in the upper-half z-plane on the semi-ellipse

$$\left(\frac{ku}{k^2+1}\right)^2+\left(\frac{kv}{k^2-1}\right)^2=1,$$ in the upper-half w-plane.

35. Find the transformation which maps the outside of the ellipse $|z-2|+|z+2|=\frac{10}{7}$ on the circle $|w|\leqslant 1$.

36. Show that $w=-\frac{1}{2}\left(z+\frac{1}{z}\right)$ maps the upper half of the circle $|z|<1$ on the upper half of the w-plane. At what points of the z-plane is the linear magnification equal to $\frac{1}{2}$? At what points is the rotation equal to $\pm\frac{1}{2}\pi$? Prove that the magnification is greater than unity throughout the interior of the semi-circle $3|z|^2=1$ in the upper half-plane.

37. Prove that, by the transformation $w=\frac{c}{2}\left(\frac{z-a}{z+a}+\frac{z+a}{z-a}\right)$ two sets of coaxal circles are transformed into sets of confocal conics. What region of the w-plane corresponds to the inside of the circle $|(z-a)/(z+a)|=\frac{1}{2}$?

38. Show that $w=\frac{2z}{1-z^2}$ maps two of the four domains, into which the circles $|z-1|=\sqrt{2}$, $|z+1|=\sqrt{2}$ divide the z-plane conformally on $|w|<1$.

39. Show that if the relation between z and w planes be given by

$$z^2+2zw\cos\lambda+w^2=1,$$

and if z describes an ellipse whose foci are the branch points in the z-plane, w describes an ellipse whose foci are the branch points in the w-plane. (*Agra 1946*)

40. Use the transformations $\zeta=\sqrt{z}$, $t=\sin(\frac{1}{2}\pi\zeta)$.

and $w=\frac{t-1}{t-1}$ to show that $w=\frac{\sin(\frac{1}{2}\pi\sqrt{z})-1}{\sin(\frac{1}{2}\pi\sqrt{z})+1}$

maps the inside of the parabola $r=\frac{2}{1+\cos\theta}$ in the z-plane cut from the focus $(z=0)$ to the point $z=-\infty$, on the unit circle in the w-plane cut from $w=0$ to $w=1$.

ANSWERS

1. Writing the given transformation as a combination of

(i) $z_1=z-4$, (ii) $z_2=\frac{1}{z_1}$, (iii) $w=11\ z_2+2$,

we see that (i) transforms the given circle into a circle passing through the origin which is transformed into a straight line by (ii). This st. line is preserved by (iii).

2. $w=i\frac{1-z}{1+z}$.

4. $w=i\frac{2-z}{z}$; $|z|\leqslant 1$ maps on $|w+i|\geqslant 2$, $\mathrm{Re}\,(z)>0$ on $\mathrm{Im}\,(w)|>-1$, $\mathrm{Im}\,(z)>0$, on $\mathrm{Re}\,(w)>0$, $\mathrm{Re}\,(z)>1$ on $|w|<1$, $1<|z|<2$ on $1<|w+i|<2$ etc.

5. $w=\frac{1-z}{2+z}$; yes.

6. Real axis between $z=-\infty$ and $z=-1$ is mapped on the arc of the circle $|w|=1$ in the second quadrant and between $z=1$ and $z=\infty$ on the arc of $|w|=1$ in the first quadrant.

7. $\frac{w-a}{w-b}=\frac{-4+3i}{5}\cdot\frac{z-a}{z-b}$, where $a=4+3i$ and $b=-4+3i$.

9. $w=\frac{z-a}{z-\bar{a}}\,e^{i\lambda}$, λ real and $\mathrm{Im}\,(a)>0$.

11. (i) $|w-1|\geqslant\frac{2}{r}$ corresponds to $|z+i|\leqslant r$.

(ii) The region bounded by the circles $|w-1|=1$, $|w|=1$ and the negative imaginary axis.

14. $w=\frac{z+i}{z-i}$; (i) Circles with centres on the imaginary axis and passing through the points $w=1$, $w=-1$; (ii) The circles $\left|\frac{w+1}{w-1}\right|=r\ (r>1)$.

16. Lower half of the circle $|z|=c$.

21. Lower-half z-plane.

22. $\zeta = ia\dfrac{1-ic\cot z/2}{1+ic\cot z/2}$.

24. $2b^2e^{2a}\left[1+\dfrac{(2b)^2}{\lfloor 3}+\dfrac{(2b)^4}{\lfloor 5}+\cdots\right]$.

35. $z=\dfrac{w}{7}+\dfrac{7}{w}$.

36. At points on the rectangular hyperbola $2x^2-2y^2=1$; at points where $r^2=\cos 2\theta$, where $w=re^{i\theta}$.

37. Exterior of the ellipse $\dfrac{16u^2}{25c^2}+\dfrac{16v^2}{9c^2}=1$.

CHAPTER IV

COMPLEX INTEGRATION AND POWER SERIES

4.1. Complex Integration.

In real analysis, we define indefinite integration as an operation inverse to differentiation and definite integration as the limit of a process of summation. The two definitions give the same result and this fact is a very important and fundamental result of the calculus.

No difficulty is experienced in extending the concept of indefinite integration to analytic functions of a complex variable. Thus, the indefinite integral $F(z)$ of an analytic function $f(z)$ is defined by the relation $F'(z)=f(z)$. The notation $\int f(z)dz$ for indefinite integral $F(z)$ of $f(z)$ is explained in the articles that follow.

However, the extension of the concept of definite integration to the complex domain is not so easy. In the case of a definite integral of a real function $\int_a^b f(x)dx$, the path of integration is always along the segment of the x-axis between the point $(a, 0)$ and $(b,0)$. In the case of a definite integral of a complex function $\int_{z_o}^{z_n} f(z)dz$, ...(1)

on the other hand, the path of integration can be any curve from z_o to z_n. The integral (1) is called a *line integral* (or, a *curvilinear integral*) over the curve. Its value might, therefore, be expected to depend upon the path of integration selected. We shall show that this is not the case when the path of integration is a regular curve. The case when the definite integral (1) is independent of the path of integration is the most important one.

Let the equations to a regular curve C joining the points z_o and z_n be $x=\phi(t)$, $y=\psi(t)$, $t_o \leqslant t \leqslant t_n$, so that the point z moves along the curve steadily from z_o to z_n as t varies from t_o to t_n. If $f(z)$ is a one-valued continuous function of z, then u and v are one-valued continuous functions of x and y, and therefore of t, when the point z is on C. Let $z_o, z_1, z_2, \ldots z_{n-1}, z_n$ be the points on C which correspond to the values $t_o, t_1, t_2, \ldots, t_{n-1}, t_n$ of the parameter t, where $t_o < t_1 < t_2 < \ldots < t_{n-1} < t_n$.

Consider the sum $\sum_{r=1}^{n} f(\zeta_r)\,(z_r - z_{r-1})$, ...(2)

where ζ_r is a point of C between z_{r-1} and z_r. If the sum (2) tends to a definite limit J as $n \to \infty$ and the maximum of the numbers $t_r - t_{r-1}$ tends to zero, we say that, **f(z) is integrable from z_o to z_n along the curve C,** and we write $J = \int_C f(x)dx$.

This definition is due to Riemann.

As t increases, the point $x(t)+iy(t)$ moves from z_o to z_n; hence the direction of integration is from z_o to z_n.

Further, writing $\zeta_r = \xi_r + i\eta_r$, $u_r = u(\xi_r, \eta_r)$, $v_r = v(\xi_r, \eta_r)$, where $f(z) = u(x, y) + iv(x, y)$, the sum (2) is

$$\sum_{r=1}^{n} \{(u_r + iv_r)\,(x_r + iy_r - x_{r-1} - iy_{r-1})\}.$$

Now, by mean-value theorem, $x_r - x_{r-1} = \phi(t_r) - \phi(t_{r-1}) = (t_r - t_{r-1})\,\phi'(t_r)$, where $t_{r-1} \leqslant T_r \leqslant t_r$.

Similarly, $y_r - y_{x-1} = (t_r - t_{r-1})\psi'(T'_r)$, where $t_{r-1} \leqslant T'_r \leqslant t_r$.

Hence, the sum may be written

$$\sum_{r=1}^{n} [(u_r + iv_r)\,\{\phi'(T_r) + i\psi'(T'_r)\}\,(t_r - t_{r-1})] \qquad \ldots\ldots(3)$$

Since all the functions concerned are continuous (and therefore uniformly continuous), we can, given ε, find δ so that

$| u_r\phi'(T_r) - u(x_r, y_r)\phi'(t_r) | < \varepsilon$ for every r, provided that each $| t_r - t_{r-1} | < \delta$.

Also $\sum_{r=1}^{n} \{\varepsilon(t_r - t_{r-1})\} = \varepsilon(t_n - t_o)$.

It follows that, as ε and $\delta \to 0$. $\sum_{r=1}^{n} \{u_r \phi'(T_r)\,(t_r - t_{r-1})\}$ tends to the same limit as $\sum_{r=1}^{n} \{u(x_r, y_r)\phi'(t_r)(t_r - t_{r-1})\}$, which, according to the definition of the definite integral of a continuous real function, tends to the limit $\int_{t_o}^{t_n} u\{\phi(t), \psi(t)\}\, \phi'(t)dt$.

Similarly, the other terms of (3) tend to limits, and we find that the whole sum tends to the limit $\int_{t_o}^{t_n} (u+iv)\,\{\phi'(t)+i\psi'(t)\}\,dt$, ...(4) the integral being interpreted in the obvious way as the sum of two real line integrals, one of which is multiplied by i.

This limit (4) is taken as the definition of the complex integral (1) along C, and we write $\int_C f(z)dz = \int_{t_o}^{t_n} (u+iv)\,\{\phi'(t)+i\psi'(t)\}dt$.(5)

This representation of the complex integral in terms of real integrals shows that we can replace dz by $dx+idy$ and expand the integrand,

$$i.e., \int_C f(z)dz = \int_C (u+iv)\,(dx+idy).$$

$$= \int_C (udx - vdy) + i \int_C (vdx + udy). \qquad(6)$$

4.11. Contours. The regular curve C along which the integral has been taken (§4·1) is called the **contour** of integration. In general, a **contour** is defined as a continuous chain of a finite number of regular arcs. A simple closed curve which consists of a finite number of regular arcs is called a **closed** contour, *e.g.*, a circle, an ellipse, a polygon etc. If $z = z_o$ in §4·1, we get a closed contour.

When the contour C is closed, the $\int_C f(z)dz$ does not indicate the direction in which the contour is described. By convention, the positive sense of description of a closed contour is taken to be

counterclockwise and we shall always assume it to be so unless indicated otherwise. The notation $\oint_C f(z)dz$ is also used to indicate the sense of description of the closed contour.

When the contour C consists of a finite number of regular arcs C_r $(r=1,2,3,\ldots, k)$, then $\int_C f(z)dz=\sum_{r=1}^{k} \int_{C_r} f(z)dz$.

4.12. Elementary Properties of Complex Integrals.

We have seen (§4·1) that the complex integral is simply a combination of two real line integrals; hence, some of the most obvious properties of real integrals extend at once to complex integrals.

For example $\int_C \{f(z)+\phi(z)\}dz=\int_C f(z)dz+\int_C \phi(z)dz$,

and $\int_C kf(z)dz=k\int_C f(z)dz$, where k is a constant.

Also, if C_1 denotes the contour C described in the opposite direction,

$$\int_{C_1} f(z)dz=-\int_C f(z)\,dz.$$

Ex. 1. *Find the value of the integral of* $\frac{1}{z-z_o}$ *round the circle* C *whose equation is* $|z-z_o|=r$.

The parametric equation of C is $z-z_o=re^{i\theta}$ where θ varies from 0 to 2π as z describes C once in the positive sense.

Hence, $\int_C \frac{dz}{z-z_o}=\int_0^{2\pi} \frac{1}{re^{i\theta}}.\ ire^{i\theta}\,d\theta=i\int_0^{2\pi} d\theta$

$=2\pi i$.

Ex. 2. *Prove that* $\int_C (z-z_o)^n\ dz=0$, *where* C *is the circle* $|z-z_o|=r$, *and* n *is any integer, positive or negative, other than* -1.

The parametric equation to C is $z-z_0=re^{i\theta}$, where θ varies from 0 to 2π.

Hence, $$\int_C (z-z_0)^n dz = \int_0^{2\pi} r^n e^{ni\theta} \cdot ire^{i\theta}\, d\theta = ir^{n+1}\int_0^{2\pi} e^{(n+1)\theta i}\, d\theta$$

$$= \frac{r^{n+1}}{n+1} \Big| e^{(n+1)\theta i} \Big|_0^{2\pi}, \quad (\text{if } n \neq -1)$$

$$= 0 \quad \left[\because e^{2\pi(n+1)i} = 1.\right]$$

If $n=-1$, this is the case of the preceding example.

In particular, when $z_0=0$, $\int_C z^n dz = 0$, $(n \neq -1)$.

4.13. An Upper Bound for Complex Integrals

Theorem. **If M is the upper bound of $|f(x)|$ on a contour C of length L joining z_0 to z_n then $\left|\int_C f(z)dz\right| \leqslant ML$.**

Since the modulus or the sum of n complex numbers is not greater than the sum of their moduli [§ 1·9, Th. I], we have

$$\left|\sum_{r=1}^{n} \{f(\zeta_r)(z_r-z_{r-1})\}\right| \leqslant \sum_{r=1}^{n} \{|f(\zeta_r)| \cdot |z_r-z_{r-1}|\}$$

$$\leqslant M\sum_{r=1}^{n} |z_r-z_{r-1}|.$$

Passing to the limit as $n\to\infty$ and max. $|z_r-z_{r-1}| \to 0$, we have the relation

$$\left|\int_C f(z)dz\right| \leqslant \int_C |f(z)|\,|dz| \leqslant M\int_C |dz|.$$

Now $\int_C |dz| = \int_{t_0}^{t_n} \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}\, dt = L$, the length of the contour from z_0 to z_n which correspond to t_0 and t_n respectively.

Hence, $\left|\int_C f(z)dz\right| \leqslant ML$.

4.2. Cauchy's Theorem (*Riemann's Proof*).

If f(z) is an analytic function of z and if f′(z) is continuous at each point within and on a closed contour C, then

$$\int_C \mathbf{f(z)dz}=0.$$

Let R be the region bounded by the closed contour C and $P(x,y)$, $Q(x,y)$, $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$ be continuous functions of x and y in R then by Green's theorem, we have

$$\int_C (P\,dx+Q\,dy)=\int\int_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dx\,dy. \quad \ldots\ldots(1)$$

Now, by §4·1 (6), we have

$$\int_C f(z)\,dz=\int_C (u\,dx-v\,dy)+i\int_C (v\,dx+u\,dy)$$

$$=-\int\int_R \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) dx\,dy+i\int\int_R \left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) dx\,dy, \quad \ldots\ldots(2)$$

since $f'(z)$ and, therefore, u_x, v_x, u_y, v_y are continuous in R, thus satisfying the conditions of Green's theorem.

In view of the Cauchy-Riemann conditions, the integrands of the two double integrals vanish throughout the region R.

Hence, $\int_C f(z)dz=0.$

Note. This is the simplest and original form of Cauchy's theorem. The proof which depends on the two-dimensional form of Green's theorem requires that $f'(z)$ be continuous. However, Goursat has shown that this assumption of continuity of $f'(z)$ is not necessary. The removal of this condition is of prime importance to the theory of analytic functions. In fact there is no justification for assuming that the derivative $f'(z)$ of an analytic function $f(z)$ is continuous. This is one of the results to be deduced from Cauchy's theorem.

4.21. Cauchy's Theorem (*Goursat's Proof*)

If a function f(z) is analytic at all points within and on a closed contour C, then $\int_C \mathbf{f(z)dz}=0$.

The proof depends on the following lemma.

Lemma. *Given any arbitrarily small positive number* ε, *it is always possible to divide the interior of C into a finite number of meshes, either complete squares or partial squares, such that, there exists a point* z_0 *within each mesh such that* $\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right|<\varepsilon$ *for all values of z in the mesh.*(1)

Let the interior of C be divided into a finite number of equal squares, complete or partial, by lines parallel to the coordinate axes. Now suppose that the lemma is false for at least one such mesh.

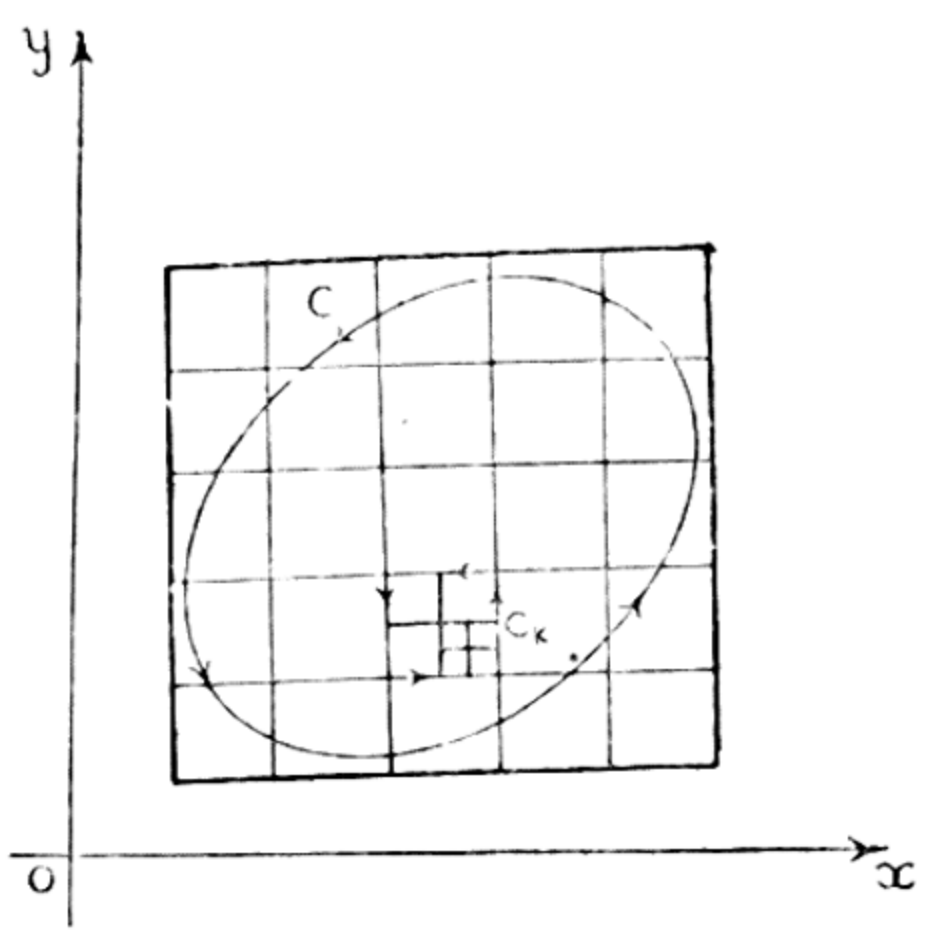


Fig. 21

Subdivide this mesh into four equal squares, complete or partial, by lines parallel to the axes. Then, in at least one of these smaller meshes, the lemma is false. Continuing this unending sequence of subdivision, the meshes, in which the lemma is false, determine a limiting point ζ which obviously lies within C. Since the lemma is false at the point ζ, $\left|\frac{f(z)-f(\zeta)}{z-\zeta}-f'(\zeta)\right|\not<\varepsilon$, where $|z-\zeta|$ is small. This shows that $f(z)$ is not differentiable at $z=\zeta$, thus contradicting the hypothesis that $f(z)$ is analytic at all points within and on the contour C. Hence, the lemma is true. Taking ζ to be z_0, the inequality (1) is satisfied.

Proof of the theorem.

Let C_k $(k=1, 2, 3,\ldots, n)$ be the boundaries of a set of meshes into which the interior of C can be divided, as in the lemma above, so that the points z_k exist for which the inequality (1) is satisfied.

The inequality can be stated in the following form :

$$\frac{f(z)-f(z_k)}{z-z_k}-f'(z_k)=\varepsilon_k, \quad \ldots(2), \text{ where } |\varepsilon_k|<\varepsilon.$$

Now let z represent any point on the boundary C_k ; then, according to equation (2),

$$f(z)=f(z_k)-z_k f'(z_k)+f'(z_k).\, z+(z-z_k)\varepsilon_k. \qquad \ldots\ldots(3)$$

Since $\int_C dz = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \{(z_r-z_{r-1}).1\}=0$, as $\max |z_r-z_{r-1}| \to 0$.

and $$\int_C z\,dz = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \{z_r(z_r-z_{r-1})\} = \text{Lt} \sum \{z_{r-1}(z_r-z_{r-1})\}$$

$$= \tfrac{1}{2}\text{Lt} \sum \{(z_r+z_{r-1})(z_r-z_{r-1})\}$$

$$= \tfrac{1}{2}\ \text{Lt}\ \Sigma(z^2_r-z^2_{r-1})=0\ ;$$

$$\therefore \quad \int_{C_k} f(z)\,dz = \int_{C_k} (z-z_k)\varepsilon_k\,dz.$$

Let the integral round each C_k be taken in the counterclockwise sense, then

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = \sum \int_{C_k} (z-z_k)\varepsilon_k\,dz$$

and hence, by § 4.13

$$\left|\int_C f(z)dz\right| \leqslant \sum \left|\int_{C_k} (z-z_k)\varepsilon_k\,dz\right| \leqslant \sum \int_{C_k} |z-z_k|\,|\varepsilon_k|\,|dz|$$

$$< \varepsilon \sum \int_{C_k} |z-z_k|\,|dz|. \qquad \ldots\ldots(4)$$

Each mesh bounded by C_k is either a complete square or a partial square. In either case let a be the side of that square. Now, $|z-z_k| < a\sqrt{2}$ (the length of the diagonal) and $\int_{C_k} |dz| = 4a$, if C_k be a square ;

$$\therefore \quad \int_{C_k} |z-z_k|\,|dz| < 4a^2\sqrt{2}, \text{ i.e., } < 4\sqrt{2}\,A_k, \ldots\ldots(5)$$

where A_k is the area of the square C_k.

If C_k is a partial square and L_k is the length of the arc of C

that forms a part of C_k, then $\int_{C_k} |z-z_k|\,|dz| < a\sqrt{2}(4a+L_k)$

$$< 4\sqrt{2}A_k + a\sqrt{2}L_k. \qquad \ldots\ldots(6)$$

If b be the side of some square containing C and A its area, and if L be the length of C, it follows from (4), (5) and (6), that

$$\left| \int_C f(z)dz \right| < \varepsilon(4\sqrt{2}A + b\sqrt{2}L), \text{ i.e., } < B\varepsilon,$$

where B is a constant. Since, ε is an arbitrarily small positive number, it follows that $\int_C f(z)dz = 0$.

Note 1. The revised form of the theorem in which it is proved above is also known as **Cauchy-Goursat theorem.**

Note 2. Cauchy's theorem is also stated in the following form :

If $f(z)$ be analytic in a region R bounded by a contour C and if C_1 and C_2 be two different contours, entirely contained in R and joining two points z_o, z_n of R, then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)dz$.

For, reversing the direction of C_2 (see fig.), the two contours C_1 and C_2 form a closed contour.

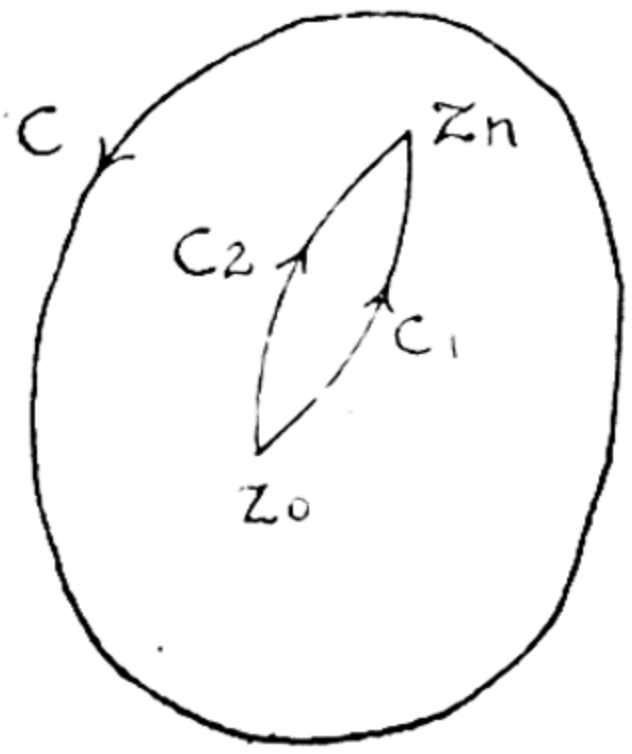


Fig. 22

Hence, $\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$.

This theorem shows the usefulness of complex integration as it establishes the independence of the integral of the particular path along which it is evaluated.

4.22. An Important Remark. *It is not necessary for the truth of Cauchy's theorem that $f(z)$ should be analytic at points on a contour closed C. The theorem holds, even if $f(z)$ is analytic in the interior of C provided it is continuous upto and on C.*

A rigorous proof of this was first given by *S.* Pollard and is beyond the scope of this book. The following explanation in intuitive language can, however, be given.

Since $f(z)$ is continuous in the interior of C, we have

$$\int_C f(z)\,dz = \mathrm{Lt} \int_{C'} f(z)dz.$$

where C' is a contour which tends to C from inside it. As $\int_{C'} f(z)dz$ vanishes for all positions of C', the result follows.

4.23. **Multiply-Connected Regions.**

A region bounded by a simple closed Jordan curve is called *simply-connected.* In general, if any simple closed Jordan curve C which consists of the points of a region R be such that all the points interior to C are also the points of R, then the region R is simply-connected.

A region whose boundary consists of two or more simple closed Jordan curves having no points in common is called *multipyl-connected.* Thus, if $C_1, C_2, \ldots C_{n-1}$ be $(n-1)$ simple closed Jordan curves having no points in common and all lying entirely within another simple closed Jordan curve C, then the multiply-connected region whose boundary consists of $C, C_1, C_2, C_3, \ldots C_{n-1}$ is said to be *of connectivity n.*

For example, the circular ring $1 < |z| < 3$ is evidently of connectivity 2, *i.e.*, the circular ring in *doubly-connected.*

The region enclosed by the closed contours C_1 or C_2 or C_3 in fig. 23 is simply-connected but the region whose boundary consists of the closed contours C, C_1, C_2 and C_3 is multiply-connected.

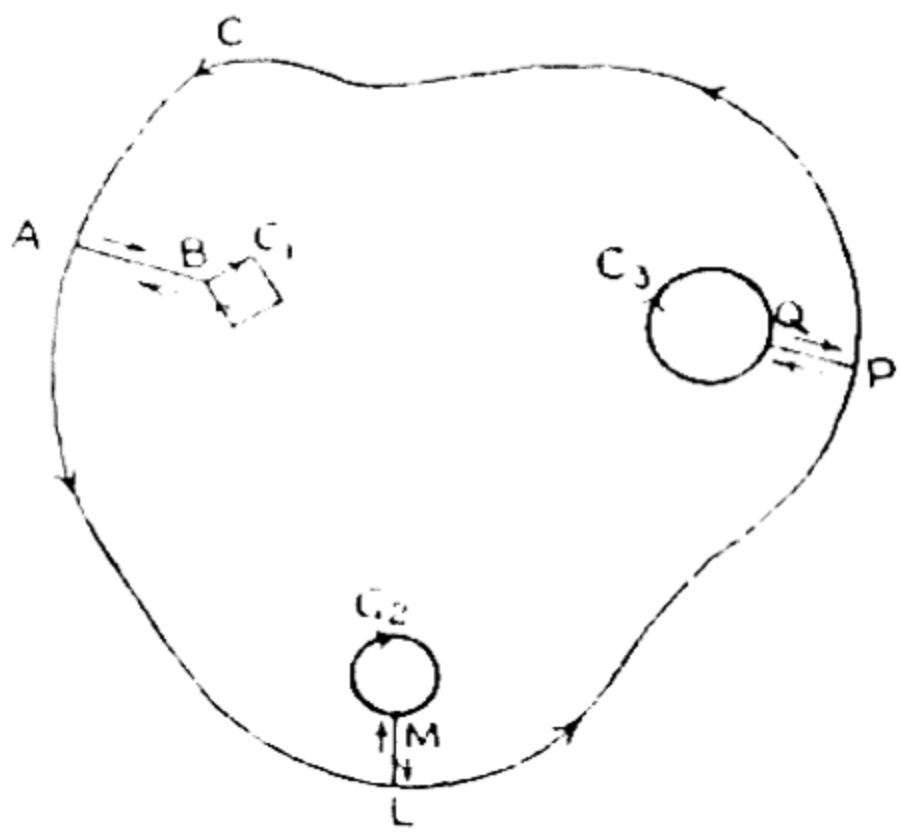


Fig. 23

A multiply-connected region can be made simply-connected by introducing cross-cuts joining the inner curve to the outer curve (see fig. 23). The boundary consisting of the curves C, C_1, C_2, C_3 and the lines AB, LM, PQ form a closed contour and Cauchy's theorem applies to such a region.

4.24. General Statement of Cauchy's Theorem.

If $f(z)$ is analytic within and on the boundary B of a multiply-connected region R, where B consists of a finite number of disconnected closed contours each described in such a sense that points of R are on the left, then

$$\int_C f(z)dz=0. \qquad \ldots\ldots(1)$$

The theorem is proved by changing the multiply-connected region into a simply-connected one by means of cuts AB, LM, PQ etc. as in fig. 23. The path is indicated by arrows in the fig. and since along a cross-cut, say, AB, the integral is taken once along AB and then along BA, the integrals along cross-cuts cancel, and we get

$$\int_C f(z)dz-\int_{C_1} f(z)dz-\int_{C_2} f(z)dz-\int_{C_3} f(z)mdz-\ldots=0,$$

since the integrals along the inner curves C_1, C_2 etc. are taken in the clockwise direction.

It follows that

$$\int_C f(z)dz=\int_{C_1} f(z)dz+\int_{C_2} f(z)dz+\int_{C_3} f(z)dz+\ \ldots, \qquad \ldots\ldots(2)$$

where all the integrals are taken in the positive sense, *i.e.*, the counter-clockwise direction.

Note. When a closed contour C_1 lies entirely within another closed contour C_2, the region between the two curves is called an *annular region.* Also

$$\int_{C_2} f(z)dz=\int_{C_1} f(z)dz \qquad \ldots\ldots(3)$$

This freedom in the choice of contours gives a great flexibility to complex integrals.

As a particular case C_1 can be taken to be a circle, Thus, the result of Ex. 1 (§ 4,12) is true for any closed contour C_1 containing the point z_o in its interior, *i.e.*, $\int_{C_1} \frac{dz}{z-z_o}=2\pi i$, since it is always possible to find r such that circle $|z-z_o|=r$ lies entirely within C_1 so that

$$\int_{C_1} \frac{dz}{z-z_o}=\int_{|z-z_0|=r} \frac{dz}{z-z_o}$$

4.25. Indefinite Integrals.

Let $f(z)$ be analytic in a simply-connected region R, then the value of the integral

$$\int_{z_0}^{z} f(Z)dZ \qquad \ldots\ldots(1)$$

depends on z only ; for, if C_1 and C_2 are two regular arcs from z_0 to z and lying entirely within R,

$$\int_{C_1} f(Z)dZ = \int_{C_2} f(Z)dZ, \qquad \ldots\ldots(2)$$

where the points on the curve are denoted by z [see § 4.21, Note 2.].

Hence, when z_0 is fixed, the integral (1) is a one-valued function $F(z)$ for all regular paths in the simply-connected region R. Such a function of the upper limit is called an *indefinite integral* or an *antiderivative* of $f(z)$.

We shall now show that the derivative of $F(z)$ is $f(z)$.

We have

$$F(z+\triangle z) - F(z) = \int_{z}^{z+\triangle z} f(Z)dZ.$$

Now, let the path of integration from z to $z+\triangle z$ be a straight line, then

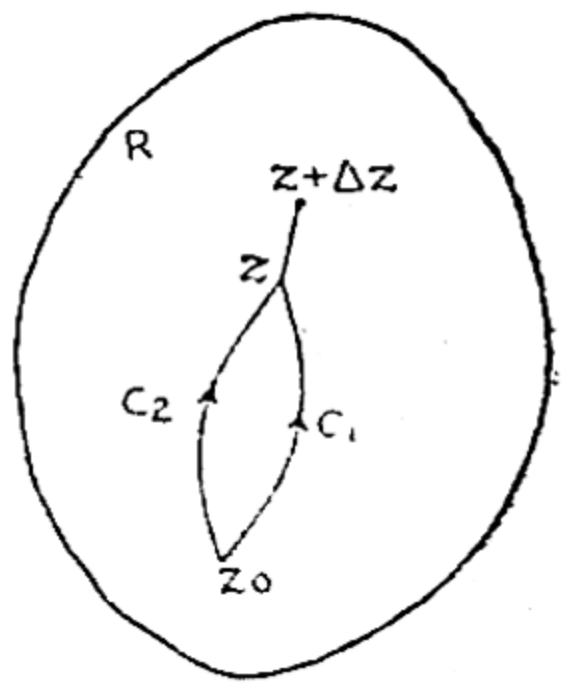


Fig. 24

$$\frac{F(z+\triangle z)-F(z)}{\triangle z} - f(z)$$

$$= \frac{1}{\triangle z}\int_{z}^{z+\triangle z} f(Z)dZ - f(z)$$

$$= \frac{1}{\triangle z}\int_{z}^{z+\triangle z} [f(Z)-f(z)]\,dz,$$

since $\frac{1}{\triangle z}\int_{z}^{z+\triangle z} f(Z)dZ = \frac{f(z)}{\triangle z}\int_{z}^{z+\triangle z} dZ = f(z).$

Also $f(z)$ is continuous; $\therefore$ $|f(Z)-f(z)| < \varepsilon$, when $|Z-z| < \delta$, or in particular, when $|\triangle z| < \delta$.

Hence, when $|\triangle z|<\delta$, we have

$$\left|\frac{F(z+\triangle z)-F(z)}{\triangle z}-f(z)\right|$$

$$\leqslant \frac{1}{|\triangle z|}\int_{z}^{z+\triangle z}|f(Z)-f(z)|\,|dz|$$

$$<\frac{1}{|\triangle z|}.\,\varepsilon|\triangle z|=\varepsilon. \qquad \text{[By § 4.13]}$$

This proves that the derivative of $F(z)$ exists, *i.e.*, $F'(z)=f(z)$ at each point z in R and, therefore, $F(z)$ is analytic.

Thus, $F(z)$, the indefinite integral of $f(z)$ is an analytic function of its upper limit.

If $G(z)$ is another analytic function, such that $G'(z)=f(z)$ throughout R ; then $\frac{d}{dz}[F(z)-G(z)]=0$.

Let $w=F(z)-G(z)$; then $\frac{dw}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$,

and, therefore, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=0$. Since w is an analytic function, we have (in view of Cauchy-Riemann conditions) $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}=0$.

Hence, u and v are constant, *i.e.*, $F(z)-G(z)$ is constant.

Let $F(z)-G(z)=k$, a constant. On putting $z=z_o$, we get

$k=-G(z_o)$, since $F(z_o)=0$.

Hence, $F(z)-G(z)=-G(z_o)$, *i.e.*, $F(z)=G(z)-G(z_o)$;

$$\therefore \quad \int_{z_o}^{z} f(Z)dZ=G(z)-G(z_o).$$

4.3. Cauchy's Integral Formula.

If f(z) is analytic within and on closed contour C, and if $z_\circ$ is any point within C, then $f(z_\circ)=\frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_\circ}$.

Let C_0 be a small circle with centre z_0 and radius r_0 so that it lies entirely within C.

Since $f(z)$ is continuous at z_0, we can take r_0 so small that $|f(z)-f(z_0)|<\varepsilon$, on C_0.

Also, since $f(z)/(z-z_0)$ is analytic in the annular region (or simply *annulus*, or ring-shaped region) between C_0 and C_1, we have, by Cauchy's Theorem [§ 4.24(3)]

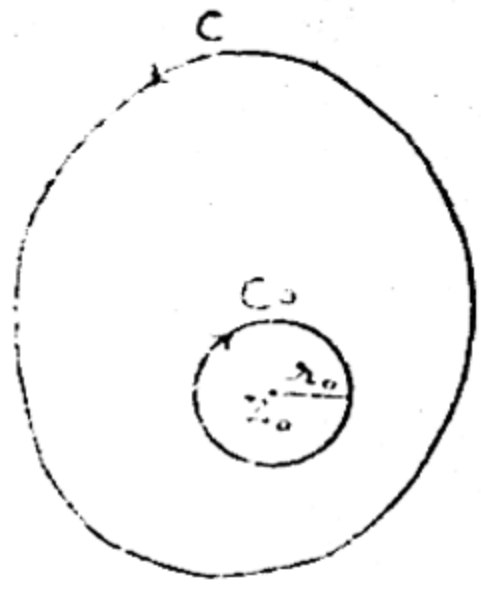


Fig. 25

$$\int_C \frac{f(z)}{z-z_0}dz=\int_{C_0}\frac{f(z)dz}{z-z_0}$$

$$=f(z_0)\int_{C_0}\frac{dz}{z-z_0}=\int_{C_0}\frac{f(z)-f(z_0)}{z-z_0}dz.$$

The first term on the right $=2\pi i\, f(z_0)$. [See Ex. 1, § 4.12]

Also, by § 4·13, $\left|\int_{C_0}\frac{f(z)-f(z_0)}{z-z_0}dz\right|\leqslant\frac{\varepsilon}{r_0}.\,2\pi r_0=2\pi\varepsilon.$

Hence, $\left|\int_C\frac{f(z)}{z-z_0}dz-2\pi i\,f(z_0)\right|\leqslant 2\pi\varepsilon.$(1)

But the L. H. S. of (1) is independent of ε. Its value must, therefore, be zero.

Thus, $\int_C\frac{f(z)}{z-z_0}dz=2\pi i\,f(z_0),$

or $f(z_0)=\frac{1}{2\pi i}\int_C\frac{f(z)dz}{z-z_0}.$(2)

Note. This is called *Cauchy's Integral Formula.* It expresses the value of $f(z)$ at any point within C in terms of an integral which depends only on the values of $f(z)$ at points on C itself. This formula is, however, applicable to functions which are analytic within and on C.

Cor. *If $f(z)$ is an analytic function in an annular region between two closed contours C and C'* (see fig. 26), *and z_0 is a point in the region, then*

$$f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$$

$$-\frac{1}{2\pi i}\int_{C'}\frac{f(z)}{z-z_0}dz,$$

where C is the outer contour.

Drawing a small circle $|z-z_0|=r_0$ in the annulus, we see that $\dfrac{f(z)}{z-z_0}$ is analytic in the region bounded by the three contours C, C' and C_0, and, therefore, by § 4·24

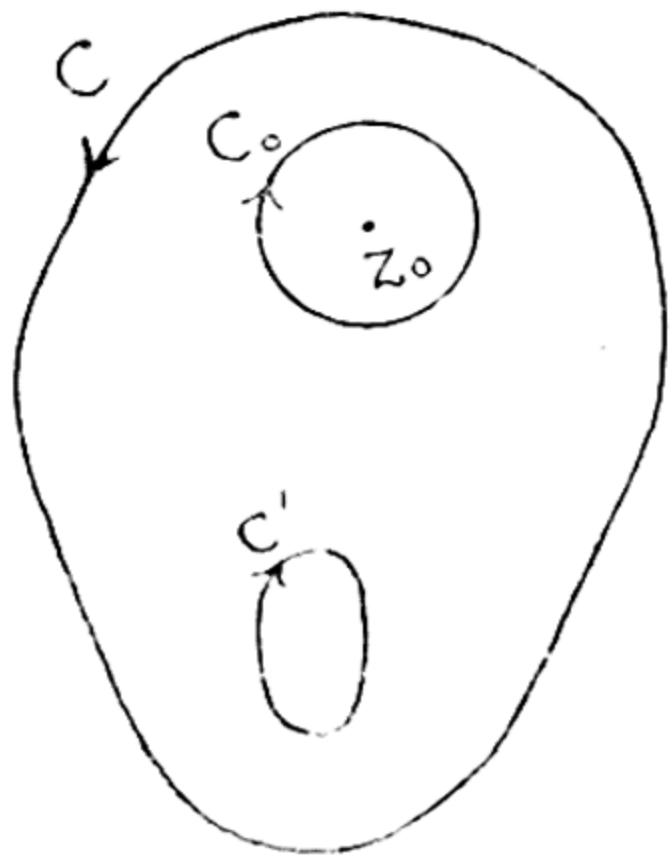


Fig. 26

$$\int_C \frac{f(z)dz}{z-z_0}=\int_{C'}\frac{f(z)dz}{z-z_0}+\int_{C_0}\frac{f(z)dz}{z-z_0},$$

from which the result follows.

4.31. Derivatives of Analytic Functions.

If f(z) is analytic in a simply-connected region R, it possesses a derivative which is again an analytic function in R. At any point $z=z_0$ within R this derivative is given by $f'(z_0)=\dfrac{1}{2\pi i}\int_C \dfrac{f(z)}{(z-z_0)^2}dz$, where C is any closed contour in R surrounding the point z_0.

Let $z_0+\triangle z_0$ be a point in C in the neighbourhood of z_0, then, since $f(z)$ is analytic within and on the contour C, we have, by Cauchy's Integral formula

$$\frac{f(z_0+\triangle z_0)-f(z_0)}{\triangle z_0}$$

$$=\frac{1}{2\pi i}\int_C \frac{1}{\triangle z_0}\left[\frac{f(z)}{z-z_0-\triangle z_0}-\frac{f(z)}{z-z_0}\right]dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_o-\triangle z_o)(z-z_o)}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_o)^2} + \frac{\triangle z_o}{2\pi i} \int_C \frac{f(z)dz}{(z-z_o)^2(z-z_o-\triangle z_o)} \qquad \ldots\ldots(1)$$

Now, since $f(z)$ is analytic on C, it is bounded there ; let $|f(z)| \leqslant M$ on C and let L be the length of C. If δ is the shortest distance from z_o to C and if $|\triangle_o z| < \delta$, we have

$$\left| \triangle z_o \int_C \frac{f(z)\,dz}{(z-z_o)^2(z-z_o-\triangle z_o)} \right| < \frac{ML\,|\triangle z_o|}{\delta^2(\delta - |\triangle z_o|)}$$

which is bounded as $|\triangle z_o| \to 0$.

Hence, the second term on the right of (1) tends to zero with $|\triangle z_o|$, and,

$$\therefore \quad f'(z_o) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_o)^2} dz. \qquad \ldots\ldots(2)$$

This is Cauchy's formula for $f'(z_o)$.

We started with the hypothesis that $f'(z)$ exists [since $f(z)$ is taken to be analytic] but now that we have proved that $f'(z)$ exists and obtained the formula for it, we can repeat the above process.

We have

$$\frac{f'(z_o+\triangle z_o) - f'(z_o)}{\triangle z_o} = \frac{1}{2\pi i} \int_C \frac{2z-2z_o-\triangle z_o}{(z-z_o)^2(z-z_o-\triangle z_o)^2} f(z)\,dz$$

and we prove as before that when $\triangle z_o \to 0$, this tends to the limit $\frac{2}{2\pi i} \int_C \frac{f(z)dz}{(z-z_o)^3}$. Hence, $f''(z)$, the derivative of $f'(z)$ exists, and is given by the formula

$$f''(z_o) = \frac{2}{2\pi i} \int_C \frac{f(z)dz}{(z-z_o)^3}. \qquad \ldots\ldots(3)$$

The reasoning used in establishing formulae (2) and (3) can be applied repeatedly to obtain a formula for the derivative of any given order. Hence, by mathematical induction, we have the general formula

$$f^n(z_o) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_o)^{n+1}},\ (n=1,\ 2,\ 3,\ldots); \qquad \ldots\ldots(4)$$

i. e., assuming this formula to be true for $n=m$, we can show that it is also true for $n=m+1$. The details of the proof are left to the reader.

Since the derivatives of all orders, $f'(z)$, $f''(z),\ldots,$ $f^n(z),\ldots\ldots$ exist in the neighbourhood of each point z_o within C, it follows that $f'(z)$, $f''(z),\ldots,f^n(z),\ldots\ldots$ are all analytic functions in R. Thus, we can state the following fundamental result.

Theorem. *A function $f(z)$, analytic in a simply-connected region R, possesses derivatives $f^n(z)$ of all orders all of which are again analytic functions in R. If C is any closed contour in R surrounding the point z_o within R, then*

$$f^n(z_o) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)dz}{(z-z_o)^{n+1}}.$$

Note 1. It can be shown as in § 4·24 that Cauchy's integral formula (§ 4·3) as well as the formula (4) above hold, if the contour C is replaced by the boundary of a multiply-connected region containing the point z_o in its interior.

Note 2. The theorem above completes the proofs of the theorem in § 2·51 and the equation (3) § 2·6. We assumed there that if $w=u+iv$ is an analytic function, then the partial derivatives of u and v of the first and second orders exist and are continuous functions of x and y. This assumption is justified by the theorem above. The theorem above is one of the most curious facts in the theory of functions of a complex variable.

Note 3. An important deduction from the theorem above is the following converse of Cauchy's theorem, known as *Morera's theorom.*

4.32. Morera's Theorem.

If f(z) is continuous and one-valued in a simply-connected region R, and if $\int_C f(z)\,dz=0$, for every closed contour C in R, then the function f(z) is analytic in R.

If z_o is a fixed point and z a variable point, both in the interior of R, then the integral $\int_{z_o}^{z} f(Z)\ dZ$ is independent of the path

(§ 4·21, Note 2) and hence represents a one-valued function $F(z)$ for all regular paths in R.

Now, in proving that $F'(z)=f(z)$ [see § 4·25], we only assumed:

(i) the continuity of $f(z)$ in R,

and (ii) that $\int_C f(z)dz=0$ for some closed contour C in R.

Both these conditions are satisfied by $f(z)$ here. Hence, the function $F(z)$ is analytic in R, and has the derivative $f(z)$. But in § 4·31, we have shown that the derivative of an analytic function is analytic. Hence, $F'(z)$, *i.e.*, $f(z)$ is analytic.

Note. It can be shown as in § 4·24 that Morera's theorem holds if C is replaced by the boundary of a multiply-connected region.

4.33. Necessary and Sufficient Condition for f(z) to be Analytic.

If $f(z)$ is analytic in a given region, Cauchy's theorem states the necessary condition. Morera's theorem states a sufficient condition for the continuous function $f(z)$ to be analytic. Combining these two theorems, we state :

The necessary and sufficient condition that a continuous function $f(z)$ is analytic in a given region R is that $\int_C f(z)\,dz=0$, *for every closed contour C in R.*

4.34. Cauchy's Inequality.

If $f(z)$ is analytic within a circle C_o, $|z-z_o|=r_o$, and if $|f(z)|<M$ on C_o, then $|f^n(z_o)|\leq \frac{M\lfloor n}{r_o^n}$.

By § 4·31, $f^n(z_o)=\frac{\lfloor n}{2\pi i}\int_{C_o}\frac{f(z)}{(z-z_o)^{n+1}}dz$.

On C_o, $\left|\frac{f(z)}{(z-z_o)^{n+1}}\right|<\frac{M}{r_o^{n+1}}$, and the length of C_o is $2\pi r_o$;

hence, by § 4·13, $|f^n(z_o)|\leq\frac{\lfloor n}{2\pi}\int_{C_o}\left|\frac{f(z)}{(z-z_o)^{n+1}}\right||dz|$

$$\leqslant \frac{\lfloor n}{2\pi} \cdot \frac{M}{r_o^{n+1}} \cdot 2\pi r_o = \frac{M \lfloor n}{r_o^n},$$

$$i.\ e., \quad |f^n(z_o)| \leqslant \frac{M \lfloor n}{r_o^n}.$$

4.35. Liouville's Theorem. If a function f(z) is analytic in the whole z-plane and | f(z) | is bounded, then f(z) must be a constant.

Let $|f(z)| \leqslant M$ for all values of z, then Cauchy's inequality (§ 4·34), we have $|f'(z_o)| \leqslant \frac{M}{r_o}$, for every z_o and r_o. Making $r_o \to \infty$. we find that $f'(z_o)=0$ at every point z_o. Hence, $f(z)$ is a constant.

Aliter. Let z_1 and z_2 be, any two points and let C be a large circle $|z| = R$ such that z_1 and z_2 lie within C, i.e., $R > |z_1|$ as well as $|z_2|$. By Cauchy's integral formula, we have

$$f(z_1)-f(z_2)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_1}\,dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_2}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{z_1-z_2}{(z-z_1)(z-z_2)} f(z)dz\ ;$$

$$\therefore \quad |f(z_1)-f(z_2)| \leqslant \frac{MR\,|z_1-z_2|}{(R-|z_1|)(R-|z_2|)}, \text{ which} \to 0 \text{ as } R \to \infty.$$

Hence, $f(z_1)=f(z_2)$. Since this result is true for all values of z_1 and z_2, it follows that $f(z)$ is a constant.

Note 1. A function which is analytic in the whole z-plane is called an *integral* (or *entire*) *function.*

Note 2. Liouville's theorem can be stated as follows also :

A non-constant integral function is unbounded in the z-plane.

Note 3. In particular, if the integral function $f(z)$ is a polynomial of degree $n \geqslant 1$, the $f(z)$ is unbounded in the z-plane.

Ex. 1. *The function $f(z)$ is analytic when $|z| < R'$. Prove that if $|a| < R < R'$,*

$$f(a)=\frac{1}{2\pi i}\int_C \frac{R^2-a\bar{a}}{(z-a)(R^2-z\bar{a})} f(z)dz,$$

where C is the circle $|z|=R$.

Deduce Poisson's formula that, if $0<r<R$,

$$f(re^{i0})=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(0-\phi)+r^2}f(Re^{i\phi})\,d\phi.$$

(Agra, 1958)

Since $|a|<R$, the point a is interior to the circle $|z|=R$ and the point $\frac{R^2}{\bar{a}}$ is exterior to the circle $|z|=R$. Let

$$I=\frac{1}{2\pi i}\int_C\frac{R^2-a\bar{a}}{(z-a)(R^2-z\bar{a})}f(z)\,dz$$

$$=\frac{1}{2\pi i}\int_C\left(\frac{1}{z-a}+\frac{\bar{a}}{R^2-z\bar{a}}\right)f(z)\,dz$$

$$=\frac{1}{2\pi i}\left[\int_C\frac{f(z)}{z-a}\,dz-\bar{a}\int_C\frac{f(z)}{R^2-z\bar{a}}dz\right].$$

Now $\frac{f(z)}{R^2-z\bar{a}}$ is analytic within and on C; $\therefore\int_C\frac{f(z)}{R^2-z\bar{a}}dz=0$.

Hence, $I=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-a}dz=f(a)$, ...(1)

by Cauchy's integral formula.

Putting $a=re^{i0}$, so that $r=|a|<R$, and $z=Re^{i0}$ in the relation (1), we get

$$f(re^{i0})=\frac{1}{2\pi i}\int_0^{2\pi}\frac{(R^2-r^2)\,f\,(Re^{i\phi})}{(Re^{i\phi}-re^{i0})\{R^2-Rre^{i(\phi-0)}\}}\cdot iRe^{i\phi}d\phi$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)\,f(Re^{i\phi})\,d\phi}{\{R-re^{i(0-\phi)}\}\{R-re^{-i(0-\phi)}\}}$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)\,f(Re^{i\phi})\,d\phi}{R^2-2Rr\cos(0-\phi)+r^2}.$$

Ex. 2. *Use Liouville's theorem to prove that the equation*

$F(z)=a_0+a_1z+a_2z^2+\ldots a_kz^k=0$, $(k=1, 2, 3, \ldots; a_k\neq0)$

has at least one root.

If $F(z) \neq O$ for any z, then $f(z) = \dfrac{1}{F(z)}$ is analytic everywhere.

Since $|f(z)| \to 0$ as $|z| \to \infty$, $|f(z)|$ is bounded for all z. Hence, $f(z)$ is a constant, by Liouville's theorem. But this contradicts the hypothesis that $F(z)$ and, therefore, $f(z)$ is not a constant, since $a_k \neq 0$ $(k = 1, 2, 3, \ldots)$.

Hence, $F(z) = 0$ for at least one z.

Note. The result proved above is known as the **Fundamental Theorem of Algebra.** By repeated application of this theorem, we can show that the equation $F(z) = 0$ has exactly k roots.

4.4. Series of Complex Terms. It is presumed that the reader is acquainted with the theory of infinite series with constant complex terms. In this and the following articles, we give a brief discussion of series with variable terms.

Let the series be $\sum_{n=1}^{\infty} u_n(z) = u_1(z) + u_2(z) + u_3(z) + \ldots$. Consider the set of points z which belong to the regions of definition of all the functions $u_1(z)$, $u_2(z)$, $u_3(z)$ etc. At such a point z, the series $\sum u_n(z)$ may or may not converge. The set of points z for which every term is defined and for which the series is convergent is called the *region of convergence*.

The convergence of a power series is one of the simplest cases. we have already seen (§ 2·7) that the region of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is the circle $|z| = R$, called the circle of convergence, where $\dfrac{1}{R} = \underset{n \to \infty}{\text{Lt}} |a_n|^{\frac{1}{n}}$, and that the power series represents an analytic function within its circle of convergence provided R is a non-zero finite number.

The general problem, however, of a series representing an analytic function depends on the concept of uniform convergence which is discussed in the following article.

4.41. Uniform Convergence. The concept of *uniform convergence* of a series with variable complex terms is defined exactly in the same way as with respect to a series with variable real terms.

Thus, *the series* $\sum_1^\infty u_n(z)$ *converges uniformly in a region D if, given* $\varepsilon>0$, *there exists an integer* $N=N(\varepsilon)$ (*depending on* ε *but not on* ε) *such that*

$$| u_{n+1}(z)+u_{n+2}(z)+\ldots+u_{n+p}(z, | <\varepsilon,$$

for all $n \geqslant N$, *all* $p \geqslant 1$ *and all* z *in* D.

We give below some useful theorems on series of complex terms which can be proved exactly in the same way as the corresponding theorems for series of real terms.

I. *If* $| z | =R>0$ *is the circle of convergence of the power series* $\sum a^n z_n$, *then the series is uniformly convergent in the circle* $| z | =r$, *where* $0 < r < R$.

II. **Weierstrass's M-Test.** *The series* $\sum u_n(z)$ *is uniformly and absolutely convergent in a region D, if* $| u_n(z) | \leqslant M_n$ $n=(1, 2, 3\ldots)$ *for all* z *in the region D, where* $\sum M_n$ *is a convergent series of positive constant numbers.*

III. *The sum of a series of continuous functions of* z, *which converges uniformly in a region D, is also a continuous function of* z *in* D.

IV. *A uniformly convergent series of continuous functions can be integrated term by term, i.e., if the functions* $u_n(z)$ *are all continuous on* C *and* $\sum_1^\infty u_n(z)$ *converges uniformly on* C, *then*

$$\int_C \left\{\sum_1^\infty u_n(z)\right\} dz=\sum_1^\infty \int_C u_n(z)\, dz,$$ *where* C *is any contour in the region of convergence of the series.*

V. *If each term of a series is an analytic function, the series can be differentiated term by term at any point within a region in which the differentiated series is uniformly convergent.*

4.42. Theorem. *If the functions* $u_1(z), \ldots, u_n(z)\ldots$ *are all analytic in the same region R and if the series* $\sum_1^\infty u_n(z)$ *is uniformly convergent in R, then the sum of the series is also an analytic function in R.*

Let $S(z)=\sum_{1}^{\infty} u_n(z)$, then $S(z)$ is continuous in R, since each $u_n(z)$ is continuous [$\because u_n(z)$ is analytic] and the series is uniformly convergent.

Hence, $\int_C S(z)dz$ exists, where C is any closed contour within R. Further, because of uniform convergence, the series can be integrated term by term on C ;

$$\therefore \quad \int_C S(z)\,dz=\sum_{1}^{\infty}\int_C u_n(z)dz.$$

Now, $\int_C u_n(z)dz=0$ $(n=1,2,3,\ldots)$, by Cauchy's theorem, since every $u_n(z)$ is analytic within and on C ;

$$\therefore \quad \int_C S(z)dz=0.$$

Hence, by Morera's theorem, $S(z)$ is analytic.

4.5. Taylor's Theorem. If f(z) be analytic at all points within a circle $C_o \mid z-z_o \mid = r_o$, then at each point z inside C_o, f(z) can be expanded into the power series $\sum_{n=0}^{\infty} \frac{f^n(z_o)}{\lfloor n}(z-z_o)^n$ with a non-zero radius of convergence, and that the expansion is unique.

Since z is an arbitrary point within C_o, let $\mid z-z_o \mid = \rho < r_o$. Now, chose ρ_1 so that $\rho<\rho_1<r_o$, and let ζ be any point on the circle $C_1 \mid \zeta - z_o \mid = \rho_1$; then

$$\frac{1}{\zeta-z}=\frac{1}{(\zeta-\zeta_o)-(z-z_o)}=\frac{1}{\zeta-z_o}\left(1-\frac{z-z_o}{\zeta-z_o}\right)^{-1}$$

$$=\sum_{n=0}^{\infty}\frac{(z-z_\circ)^n}{(\zeta-z_\circ)^{n+1}}. \qquad \ldots\ldots(1)$$

The series (1) is uniformly convergent w.r.t. ζ along C_1, since

$$\left|\frac{z-z_\circ}{\zeta-z_\circ}\right|=\frac{\rho}{\rho_1}<1,$$

and the same is true for the series

$$\frac{1}{2\pi i}\cdot\frac{f(\zeta)}{\zeta-z}=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n. \qquad \ldots\ldots(2)$$

Hence, integrating the series term by term (§4.41 IV) w.r.t. ζ, we have, by §§ 4.3 and 4.31

$$f(z)=\sum_{n=0}^{\infty}\frac{f(z_0)^n}{\lfloor n}(z-z_0)^n, \qquad \ldots\ldots(3)$$

which is the desired expansion in power series. This result is known as the *Taylor's theorem* and the series on the right is called the *Taylor's series* (or *expansion*).

If possible, let $f(z)$ have a different power series expansion $\sum_0^{\infty} a_n(z-z_0)^n$ convergent in the same neighbourhood as the power series (3). We have already seen in § 2.7 that a power series can be differentiated term by term as often as we please at any point within its circle of convergence. Hence, the coefficient a_n of the series

$$f(z)=\sum a_n(z-z_0)^n, \quad |z-z|<r_0$$

can be determined by differentiated $f(z)$ n times and then putting $z=z$.

Thus $f^n(z_0)=a_n\lfloor n$, i.e., $a_n=\dfrac{f^n(z_0)}{\lfloor n}$. If follows that

$$f(z)=\sum_{n=0}^{\infty}\frac{f^n(z_0)}{\lfloor n}(z-z_0)^n.$$

Hence, every power series is a Taylor series, i.e., the expansion (3) is unique.

Note 1. This theorem is sometimes called the Cauchy-Taylor theorem, since the extension of Taylor's theorem to the domain of complex variable is due to Cauchy.

Note 2. When $f(z)$ is analytic within and on a closed contour C, the maximum radius of C_0 is the distance from z_0 to the nearest point C.

In general, the maximum radius of C_0 is the distance from the point z_0 to the nearest singular point of $f(z)$, since the function is to be analytic at all points within C_0. The circle C_0 is the circle of convergence of the Taylor series.

4.51. Remarks on Taylor's Theorem.

I. We have already seen in § 2.7 that a power series represents an analytic function inside its circle of convergence and as shown in § 4.5, a function analytic within a circle can be expanded as a power series. By combining these two results, we see that the *necessary and sufficient condition that a function $f(z)$ should have a power series expansion is that it should be analytic in a region.*

Thus, an analytic function can also be defined as a complex function which can be expanded into a convergent power series. The development of the theory of analytic functions starting from this definition is known as the Weierstrassian development.

II. The expansion of an analytic function $f(z)$ into a power series has one different feature from the corresponding expansion of a function of a real variable. In ordinary calculus, the first n terms of the expansion and a remainder term are obtained and then we investigate whether this remainder tends to zero as n tends to infinity. This investigation is not an easy task in general. In complex theory, however, it follows from the hypothesis in Taylor's Theorem that the remainder term must tend to zero.

For example, in the case of the real function $\tan x$, it is yet to be proved that the remainder after n terms tends to zero as $n \to \infty$. But, if we expand the complex function $\tan z$ into a power series in the neighbourhood of $z=0$, we see immediately that the radius of convergence of the Taylor series is $\frac{\pi}{2}$. Now, if z is real, it follows that $\tan x$ can be expanded as a power series in x if $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

4.52. Maclaurin's Theorem. Putting $z_0=0$ in Taylor's theorem [§4.5(3)], we obtain

$$f(z)=\sum_{n=0}^{\infty} \frac{f^n(0)}{\lfloor n} z^n. \qquad \ldots\ldots(1)$$

This result is known as Maclaurin's theorem.

Note. The result (1) was first discovered by Stirling (1717) and published by Maclaurin (1742) in his *Fluxions*.

Ex. 1. *Prove that*

$$\log\ z=z-1-\frac{(z-1)^2}{2}+\ldots\ldots\ldots, \qquad |\ z-1\ |<1.$$

Deduce that $|\ \log\ (1+z)\ | \leqslant -\log\ (1-|\ z\ |\),\ |\ z\ |<1.$

Let $f(z)=\log z$, then $f(z)$ is analytic within the circle $|\ z-1\ |=1$, as the only singularity is $z=0$;

$$\therefore\ f(z)=f(1+z-1)=f(1)+(z-1)f'(1)+\ldots+\frac{(z-1)^n}{\lfloor n}f^n(1)+\ldots\ldots,$$

if $|\ z-1\ |<1$.

Hence, $\log\ z=(z-1)-\frac{(z-1)^2}{2}+\ldots,\ |\ z-1\ |<1.$

$$\left[\because\ \ f^n(z)=\frac{(-1)^{n-1}}{z^n}.\right]$$

Changing z to $z+1$, we have

$$\log\ (1+z)=z-\frac{z^2}{2}+\frac{z^2}{3}-\ldots,\ |\ z\ |<1\ ;$$

$$\therefore\quad |\ \log\ (1+z)\ |\ =\ |\ z-\frac{z^2}{2}+\frac{z^3}{3}\ldots\ |$$

$$\leqslant\quad |\ z\ |+\frac{|\ z\ |^2}{2}+\frac{|\ z\ |^3}{3}+\ldots=-\log\ (1-|\ z\ |\)$$

$$[\because\ \ |\ z\ |<1.]$$

Ex. 2. The function f(z) is analytic when $|\ z\ |<R$ and has the Taylor expansion $\sum_0^\infty a_n z^n$. Show that, if $r<R$,

$$\frac{1}{2\pi}\int_0^{2\pi}|\ f(re^{i\theta})\ |^2 d\theta=\sum_0^\infty |\ a_n\ |^2 r^{2n}.$$

Hence, prove that if $|\ f(z)\ |\leqslant M$ when $|\ z\ |<R$,

$$\sum_0^\infty |\ a_n\ |^2\ r^{2n}\leqslant M^2.$$

(Agra 1945)

Let $f(z)$ be analytic within and on the circle C $|\ z\ |=r<R$, and let $\bar{a}_n$ be the conjugate of a_n ; then

$$f(z)=\sum_0^\infty a_n z^n, \text{ i.e., } f(re^{i\theta})=\sum_0^\infty a_n r^n e^{ni\theta}, \qquad \ldots\ldots(1)$$

$$\text{and} \quad \overline{f(re^{i\theta})}=\sum_0^\infty \bar{a}_m r^m e^{-mi\theta}. \qquad \ldots\ldots(2)$$

As the series (1) and (2) are both absolutely convergent, they can be multiplied term by term and the terms in the product can be re-arranged. Further, the product is uniformly convergent for $0 \leqslant 0 \leqslant 2\pi$ and can be integrated term by term over this interval, i.e., along the circle C. The integrals in which $n \neq m$ obviously vanish ; for, $\int_0^{2\pi} e^{pi\theta}\, d\theta = 0$, where p is an integer.

$$\text{Hence, we get} \quad \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_0^\infty |a_n|^2 r^{2n} \int_0^{2\pi} d\theta$$

$$= 2\pi \sum_0^\infty |a_n|^2 r^{2n}. \qquad \ldots\ldots(3)$$

If $|f(z)| \leqslant M$ on C, then

$$\sum_0^\infty |a_n|^2 r^{2n} \leqslant \frac{M^2}{2\pi} \int_0^{2\pi} d\theta = M^2. \qquad \ldots\ldots(4)$$

Note 1. The result (3) is known as *Parseval's identity*.

Nete 2. From (4), we see that

$$|a_n|^2 r^{2n} < \sum_0^\infty |a_n|^2 r^n \leqslant M^2 ;$$

$$\therefore \quad |a_n| < \frac{M}{r^n}, \text{ which is Cauchy's inequality.}$$

Further, comparing the results (3) and (4) we see that Cauchy's inequality can reduce to an equality if, and only if, all the coefficients other than a_n vanish, i.e., if $f(z)$ reduces to the from $f(z)=a_n z^n$.

Hence, Cauchy's inequality and the conditions under which it reduces to an equality, follow from the inequality (4).

4.53. Laurent's Theorem. If f(z) is analytic on and within the annulus between the circles C_1 and C_2 with centre z_0 and radii R_1 and R_2 ($R_2 < R_1$), then at any point z of the annulus, f(z) can be expanded into a series of the form

$f(z) = \sum_{n=0}^{\infty} a_n (z-z_o)^n + \sum_{n=1}^{\infty} b_n (z-z_o)^{-n}$, **convergent at all the points of the annulus, where** $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)\, d\zeta}{(\zeta-z_o)^{n+1}}$, and

$$b_n = \int_{C_2} \frac{f(\zeta)\, d\zeta}{(\zeta-z_o)^{-n+1}}.$$

Let ζ be the variable point on C_1 or C_2, then by §4.3 Cor.,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)\, d\zeta}{\zeta-z} - \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)\, d\zeta}{\zeta-z},$$

where the integrals are taken round the circles in the counter clockwise direction. In the first integral, as in the proof of Taylor's theorem (§4·5), we have $|\zeta-z_o| > |z-z_o|$;

$$\therefore \quad \frac{1}{\zeta-z} = \frac{1}{\zeta-z_o}\left(1-\frac{z-z_o}{\zeta-z_o}\right)^{-1}$$

$$= \sum_{0}^{\infty} \frac{(z-z_o)^n}{(\zeta-z_o)^{n+1}}. \qquad \ldots\ldots\ldots(1)$$

The series (1) is uniformly convergent w.r.t. ζ on C_1, since $\left|\frac{z-z_o}{\zeta-z_o}\right| < 1$, and the same is true for the series

$$\frac{1}{2\pi i}\frac{f(\zeta)}{\zeta-z} = \frac{1}{2\pi i}\sum_{0}^{\infty} \frac{f(z)}{(\zeta-z_o)^{n+1}}(z-z_o)^n. \qquad \ldots\ldots\ldots(2)$$

Hence, integrating the series term by term (§4·41 IV) w.r.t. ζ, we have

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)\, d\zeta}{\zeta-z} = \sum_{0}^{\infty} a_n (z-z_o)^n, \qquad \ldots\ldots\ldots(3)$$

where $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)\, d\zeta}{(\zeta-z_o)^{n+1}}$. $\qquad \ldots\ldots\ldots(4)$

Here $a_n \neq \frac{f^n(z_o)}{\lfloor n}$ in general, as $f(z)$ is not necessarily analytic throughout the interior of C_1.

In the second integral, we have $|\zeta - z_o| < |z - z_o|$;

$$\therefore \quad -\frac{1}{\zeta - z} = \frac{1}{z - z_o}\left(1 - \frac{\zeta - z_o}{z - z_o}\right)^{-1} = \sum_{n=1}^{\infty} \frac{(\zeta - z_o)^{n-1}}{(z - z_o)^n}. \qquad \ldots\ldots(5)$$

The series (5) is uniformly convergent w.r.t. ζ on C_2, since $\left|\frac{\zeta - z_o}{z - z_o}\right| < 1$, and the same is true for the series

$$-\frac{1}{2\pi i}\frac{f(\zeta)}{\zeta - z} = \frac{1}{2\pi i}\sum_{1}^{\infty} \frac{f(\zeta)}{(\zeta - z_o)^{-n+1}} \cdot \frac{1}{(z - z_o)^n}. \qquad \ldots\ldots(6)$$

Hence, integrating the series term by term (§ 4·41 IV) w.r.t. ζ, we have

$$-\frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)\, d\zeta}{\zeta - z} = \sum_{1}^{\infty} b_n (z - z_o)^{-n}, \qquad \ldots\ldots(7)$$

$$\text{where} \quad b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)\, d\zeta}{(\zeta - z_o)^{-n+1}}. \qquad \ldots\ldots(8)$$

It follows from equations (3) and (7), that

$$f(z) = \sum_{0}^{\infty} a_n (z - z_o)^n + \sum_{1}^{\infty} b_n (z - z_o)^{-n}, \qquad \ldots\ldots(9)$$

where a_n and b_n are given by the equations (4) and (8).

The result (9) is known as the *Laurent's theorem* and the series on the right is called the *Laurent's series* (or *expansion*) of $f(z)$.

Since the integrands of the integrals in formulae (4) and (8) are analytic functions of ζ throughout the annulus, any closed contour can be taken as the path of integration in place of the circles C_1 and C_2. In particular, if we replace the circles C_1 and C_2 by the circle $C \mid z - z_o \mid = R$, where $R_2 < R < R_1$,

we have $$b_n = \frac{1}{2\pi i}\int_{C} \frac{f(\zeta)\, d\zeta}{(\zeta - z_o)^{-n+1}} = a_{-n},$$

and Laurent's expansion of $f(z)$ is

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_o)^n, \qquad \ldots\ldots(10)$$

$$\text{where} \quad a_n = \frac{1}{2\pi i}\int_{C} \frac{f(\zeta)\, d\zeta}{(\zeta - z_o)^{n+1}} \quad (n = 0, \pm 1, \pm 2, \ldots). \qquad \ldots\ldots(11)$$

Note 1. In the particular case when $f(z)$ is analytic at all points inside C_2 all the coefficients b_n are zero (by Cauchy's theorem), and Laurent's series reduces to Taylor's series.

Note 2. Laurent's theorem gives the expansion of $f(z)$ in a series of positive and negative powers of $z-z_o$. If we consider seperately the series (3) and (7), we see that the series (3) converges everywhere inside C_1 and the series (7) converges everywhere outside C_2 and both converge at all points of the annulus.

4.54. Remark on Laurent's Theorem. We can show that like the Taylor expansion of a function, the Laurent expansion is also unique.

If possible, let $f(z)$ have a different expansion $\sum_{-\infty}^{\infty} A_n(z-z_o)^n$ convergent in the same annulus $R_2 \leqslant |z-z_o| \leqslant R_1$. Now, the coefficient a_n of the Laurent expansion $\sum_{-\infty}^{\infty} a_n(z-z_o)^n$ is given by

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_o)^{n+1}} \quad [C \text{ being the circle } |z-z_o| = R, \; R_2 < R < R_1]$$

$$= \frac{1}{2\pi i}\sum_{k=-\infty}^{\infty} A_k \int_C \frac{(z-z_o)^k}{(z-z_o)^{n+1}}\,dz,$$

on integrating term by term the series $\sum_{-\infty}^{\infty} A_n\,(z-z_o)^n$, which is uniformly convergent.

Hence, $a_n = A_n$, as all those integrals in which $k \neq n$ vanish [Ex. 2, § 4·12], and when $k=n$,

$\int_C \frac{dz}{z-z_o} = 2\pi i$ [Ex. 1, § 4·12]; i.e., the series $\sum_{-\infty}^{\infty} A_n\,(z-z_o)^n$ is identical with the Laurent series of $f(z)$.

Ex. 1. *Show that* $\sin\left\{c\left(z+\frac{1}{z}\right)\right\}$ *can be expanded in a series of the type*

$$a_o + a_1 z + a_2 z^2 + \ldots + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots,$$

in which the coefficients, both of z^n and of z^{-n}, are

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\,(2\,c\,\cos\,0)\,\cos\,n0d0.$$

(Agra 1935, '42)

The only singular point of $(z+\frac{1}{z})$ is the origin; hence, $\sin\{c\,(z+\frac{1}{z})\}$ is analytic in the annulus $r \leqslant |\,z\,| \leqslant R$, no matter how small the positive number r may be or how large R may be.

So by Laurent's theorem

$$\sin\left\{\ c\,(z+\frac{1}{z})\right\}=a_0+a_1z+a_2z^2+\cdots+\ \frac{b_1}{z}+\frac{b_2}{z^2}+\cdots,$$

$$\text{where } a_n=\frac{1}{2\pi i}\int_{C_1}\sin\left\{\ c\left(\ z+\frac{1}{z}\right)\ \right\}\frac{dz}{z^{n+1}},$$

$$\text{and}\quad b_n=\frac{1}{2\pi i}\int_{C_2}\sin\left\{\ c\left(\ z+\frac{1}{z}\right)\ \right\}z^{n-1}\,dz,$$

and where C_1 and C_2 are any circles with the origin as the centre.

Taking C_1 to be $|\,z\,|=1$ and writing $z=e^{0i}$, where $0\leqslant 0\leqslant 2\pi$, we have

$$a_n=\frac{1}{2\pi i}\int_0^{2\pi}\sin\,(2c\,\cos\,0).\ \frac{ie^{0i}\,d0}{e^{(n+1)0i}}=\frac{1}{2\pi}\int_0^{2\pi}\sin\,(2c\,\cos 0)\ \ e^{-n0i}d0$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\sin\,(2c\,\cos\,0)\,\cos\,n0\ d0,$$

since $F(0)=\int_0^{2\pi}\sin\,(2c\,\cos\,0)\,\sin\,n0\ d0=0$, as may be seen by writing $2\pi-0$ for 0. [If $F\,(2\pi-0)=-F\,(0)$, then $F\,(0)=0$.]

Now $b_n=a_n$, since the function expanded is unaltered if $\frac{1}{z}$ be written for z. Hence, $b_n=a_n=\frac{1}{2\pi}\int_0^{2\pi}\sin\,(2c\,\cos\,0)\,\cos\,n0\ d0.$

Ex. 2. *Obtain expansions for* $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ *which are valid*

(i) *when* $|z|<1$;
(ii) *when* $1<|z|<4$;
(iii) *when* $|z|>4$. *(Agra 1954)*

Let $f(z)=\frac{(z-2)(z+2)}{(z+1)(z+4)}=1-\frac{1}{z+1}-\frac{4}{z+4}$.

This function has two singular points, namely $z=-1$ and $z=-4$.

(i) *Method I.* Within the circle $|z|=1$, the function can be represented by a Maclaurin series.

Now $f(0)=-1$, and

$$\frac{f^n(z)}{\lfloor n}=(-1)^{n-1}\left[\frac{1}{(z+1)^{n+1}}+\frac{4}{(z+4)^{n+1}}\right],$$

i. e., $$\frac{f^n(0)}{\lfloor n}=(-1)^{n-1}\left(1+\frac{1}{4^n}\right).$$

Hence, when $|z|<1, f(z)=-1+\sum_{n=1}^{\infty}\{(-1)^{n-1}(1+4^{-n})z^n\}$.

.....(1)

Method II. Since, every power series expansion of $f(z)$ convergent in $|z|<1$ is the Taylor expansion, we have for $|z|<1$

$$f(z)=1-(1+z)^{-1}-\left((1+\frac{z}{4}\right)^{-1}$$

$$=1-\sum_0^{\infty}(-1)^n z^n-\sum_0^{\infty}(-1)^n\left(\frac{z}{4}\right)^n=-1+\sum_{n=1}^{\infty}\{(-1)^{n-1}(1+4^{-n})z^n\}$$

(ii) The function $f(z)$ is analytic in the annulus $1<|z|<4$. Hence, we can obtain the Laurent expansion of $f(z)$ in this annulus.

Method I Let $f(z)=\sum_{-\infty}^{\infty}a_n z^n$,

then $a_n=\frac{1}{2\pi i}\int_C\frac{f(z)}{z^n}dz$, where C is any circle between $|z|=1$ and $|z|=4$;

$$\text{i.e. } a_n=\frac{1}{2\pi i}\int_C\left\{\frac{1}{z^{n+1}}-\frac{1}{z^{n+1}}\left(\frac{1}{z+1}+\frac{4}{z+4}\right)\right\}dz$$

$$=\frac{1}{2\pi i}\left[\int_C\frac{dz}{z^{n+1}}-\int_C\left\{\frac{1}{z^{n+1}}-\frac{1}{z_n}+\ldots+-\frac{(-1)^n}{z}\right.\right.$$

$$\left.\left.+\frac{(-1)^{n+1}}{z+1}\right\}dz\right.$$

$$-\frac{1}{4^n}\int_C\left\{\frac{4^n}{z^{n+1}}-\frac{4^{n-1}}{z^n}+\ldots+\frac{(-1)^n}{z}+\frac{(-1)^{n+1}}{z+4}\right\}dz,$$

on decomposing the integrand into partial fractions.

Since $\int_C\frac{dz}{z^{n+1}}=0,\ n\neq 0$ and $\int_C\frac{dz}{z^{n+1}}=2\pi i,\ n=0$;

$$\therefore\quad a_n=\frac{(-1)^{n+1}}{2\pi i}\left[2\pi i-\int_C\frac{dz}{z+1}+\frac{1}{4^n}\left(2\pi i-\int_C\frac{dz}{z+4}\right)\right]$$

$(n\neq 0)$,

and $$a_o=\frac{1}{2\pi i}\left[\int_C\frac{dz}{z+1}-2\pi i+\int_C\frac{dz}{z+4}\right].$$

But $\frac{1}{z+4}$ is analytic within C and hence $\int_C\frac{dz}{z+4}=0.$

To evaluate the other integral, we have (by Cauchy's theorem)

$\int_C\frac{dz}{z+1}=\int_{C'}\frac{dz}{z+1}$, where C' is a small circle about the point $z=-1$ and put $z+1=\rho e^{0i}$, where ρ is a constant and $0\leqslant 0\leqslant 2\pi$.

We have then $\int_C\frac{dz}{z+1}=\int_0^{2\pi}id0=2\pi i$;

$$\therefore\quad a_n=\frac{(-1)^{n+1}}{4^n}\ (n\neq 0)\text{ and }a_o=0.$$

Again, $b_n=\frac{1}{2\pi i}\int_C z^{n-1}\left(1-\frac{1}{z+1}-\frac{4}{z+4}\right)dz$, where $n\geqslant 1$.

But $\frac{z^{n-1}}{z+4}$ is analytic within C, and hence $\int_C \frac{z^{n-1}}{z+4}dz=0$. As before, take a small circle C' about the point $z=-1$, then

$$\int_C \frac{z^{n-1}}{z+1}dz=\int_{C'} \frac{z^{n-1}\,dz}{z+1}=\int_0^{2\pi} \frac{(-1+\rho e^{\theta i})^{n-1}\, i\rho e^{\theta i}}{\rho e^{\theta i}}d\theta$$

$$= i(-1)^{n-1}\int_0^{2\pi} d\theta, \text{ since other integrals vanish.}$$

$$= (-1)^{n-1}\,2\pi i\ ;$$

$$\therefore \quad b_n=(-1)^n. \quad [\ \because \int_C z^{n-1}\,dz=0.\]$$

Hence, when $1<|z|<4$,

$$f(z)=\sum_1^\infty \frac{(-1)^{n+1}}{4^n}z^n+\sum_1^\infty (-1)^n\frac{1}{z^n}. \qquad \ldots\ldots\ldots(2)$$

Method II. Since the Laurent expansion for $1<|z|<4$ is unique, any power series expansion convergent in the annulus $1<|z|<4$ is the required expansion ;

$$\therefore \quad f(z)=1-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}-\left(1+\frac{z}{4}\right)^{-1}$$

$$=1-\frac{1}{z}\sum_0^\infty (-1)^n/z^n-\sum_0^\infty (-1)^n\left(\frac{z}{4}\right)^n \quad [\ \because \quad |z|>1 \text{ and } |z|<4.\]$$

$$=\sum_1^\infty (-1)^{n-1}\left(\frac{z}{4}\right)^n-\sum_1^\infty (-1)^{n-1}/z^n.$$

(iii) *Method I.* When $|z|>4$, put $z=\frac{1}{z'}$. The transformed function is

$$f(z)=\phi(z')=\frac{(1-2z')(1+2z')}{(1+z')\ (1+4z')}=\frac{1}{1+z'}+\frac{1}{1+4z'}-1.$$

Within the circle $|z'|=\frac{1}{4}$, $\phi(z')$ is analytic and may be expanded in a Maclaurin series, giving

$$\phi(z')=(1+z')^{-1}+(1+4z')^{-1}-1=\sum_0^\infty (-1)^n(1+4^n)z'^n-1.$$

Replacing z' by $\frac{1}{z}$, we have the required expansion when $|z|>4$,

$$f(z)=1+\sum_{1}^{\infty}(-1)^n(1+4^n)/z^n. \qquad \ldots\ldots\ldots(3)$$

Method II. The required expansion is a power series convergent for $|z|>4$;

$$\therefore \quad f(z)=1-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}-\frac{4}{z}\left(1+\frac{4}{z}\right)^{-1}$$

$$=1-\sum_{1}^{\infty}\frac{(-1)^{n-1}}{z^n}-\sum_{1}^{\infty}\frac{(-1)^{n-1}.4^n}{z^n}=1+\sum_{1}^{\infty}\frac{(-1)^n(1+4^n)}{z^n}.$$

4.6. Zeros, Singularities and Poles of a Function.

I. If $f(z)$ is analytic in the neighbourhood of $z=z_0$, then

$$f(z)=\sum_{0}^{\infty}a_n(z-z_0)^n$$

is the Taylor expansion for $|z-z_0|=\delta$, where δ is small.

If $a_0=a_1=a_2=\ldots=a_{k-1}=0$, $a_k\neq 0$, the Taylor expansion begins with a term in $(z-z_0)^k$ and $f(z)$ is said to have a **zero of order k** at $z=z_0$. The zero is said to be *simple*, *double*, *triple*,..., when $k=1, 2\ \ 3,\ldots.$

II. If $f(z)$ is analytic within a region R, except at the point $z=z_0$, the point z_0 is called an **isolated singularity** of the function. In this case, a positive number r exists such that $f(z)$ is analytic in the annulus $0<|z-z_0|\leqslant r$ and can be represented by the Laurent series

$$f(z)=\sum_{0}^{\infty}a_n(z-z_0)^n+\sum_{1}^{\infty}b_n(z-z_0)^{-n}.$$

The second term on the right is called the **principal part** of $f(z)$ at the isolated singular point $z=z_0$. The following three cases arise now :

(i) If all the coefficients b_n are zero, $z=z_0$ is called a removable singularity of $f(z)$, since $f(z)$ can be made analytic in $|z-z_0| \leqslant r_0$ by suitably defining its value at z_0. Singularities of this sort are not of much importance in the theory.

(ii) If the principal part is an infinite series, z_0 is called an **isolated essential singularity.**

(iii) If the principal part is a finite series, z_0 is called a **pole.** *If* $b_n \neq 0$, while $b_{n+1}=b_{n+2}=\ldots=0$, the pole is said to be of **order n;** or to be a *simple*, *double*, *triple*,...pole when $n=1,2,3,\ldots$. The coefficient b_1 is called the **residue** of $f(z)$ at the pole $z=z_0$.

III. *If* $f(z)$ *has a pole of order* n *at* $z=z_0$, *then* $\dfrac{1}{f(z)}$ *has a zero of order* n *at* z_0, *and conversely.*

Let $f(z)=\sum_{0}^{\infty} a_n (z-z_0)^n+\sum_{k=1}^{n} \dfrac{b_k}{(z-z_0)^k}$ in an annulus $0<|z-z_0| \leqslant r$ and $b_n \neq 0$, then $f(z)=\dfrac{1}{(z-z_0)^n}\phi(z)$, where $\phi(z)$ is analytic in $|z-z_0| \leqslant r$ and $\phi(z_0)=b_n \neq 0$.

Hence, $\psi(z)=\dfrac{1}{\phi(z)}=\dfrac{1}{(z-z_0)^n f(z)}$ is also analytic and $\psi(z_0)=0$, so that $\dfrac{1}{f(z)}=(z-z_0)^n\psi(z)$ has a pole of order n at z_0.

IV. *If* $f(z)$ *has a pole at* $z=z_0$, *then* $|f(z)| \to \infty$ *as* $z \to z_0$ *in any manner.*

Let $f(z)$ have a pole of order n at $z=z_0$, and let $\phi(z)=(z-z_0)^n f(z)$, where $\phi(z_0)=b_n \neq 0$. Since $\phi(z)$ is analytic in $|z-z_0| \leqslant r$, we can find a neighbourhood $|z-z_0|<\varepsilon$, ε an arbitrarily small positive number, of the pole in which $|\phi(z)|>\frac{1}{2}|b_n|$,

and, $\therefore \quad |f(z)|>\frac{1}{2}|b_n|\,|z-z_0|^{-n}$,

Hence, $|f(z)| \to \infty$ as $z \to z_0$ in any manner.

4.61. Zeros are Isolated. Let $f(z)$ be analytic in the neighbourhood of any zero point $z=z_0$, then $f(z)=(z-z_0)^m\,\phi(z)$, if the zero is of order m. Now, $\phi(z)$ is analytic in the neighbourhood of $z=z_0$ and $\phi(z_0) \neq 0$. We can, therefore, draw a small circle C with z_0 as

centre, within which $\phi(z)$ does not vanish. Also for $z \neq z_0$ within C, $(z-z_0)^m \neq 0$. Hence z_0 is the only point within C at which $f(z)=0$. The zero point $z=z_0$ is therefore isolated. But z_0 is any zero point of $f(z)$; hence, all zeros of $f(z)$ are isolated.

4.62. Poles are Isolated. If the one-valued function $f(z)$ has a pole of order m at $z=z_0$: it has been shown in § 4.6 III that the function $\frac{1}{f(z)}$ is analytic in the neighbourhood of z_0 and has a zero of order m at $z=z_0$.

Since the zeros are isolated (§ 4.61), it follows that the poles must also be isolated which is also obvious from the definition of a pole (§ 4·6).

4.63. Limiting Point of Zeros. If $z_1, z_2, z_3 \ldots, z_n, \ldots$ is a sequence of zeros of $f(z)$, analytic in a region R and if z_0, an interior point of R, is a limiting point of this sequence, then $f(z)$ either vanishes identically or else has an isolated essential singularity.

For, since $f(z)$ is continuous having zeros as near as we please to z_0, $f(z_0)$ must be zero. Now, z_0 cannot be a zero of $f(z)$, since the zeros are isolated (§ 4·61). Hence, $f(z)$ is identically zero. If $f(z)$ is not identically zero in R, then $f(z)$ must have a singularity at $z=z_0$. In this case the singularity is isolated but it is not a pole, since $|f(z)| \neq \infty$ as $z \to z_0$ in any manner. Thus, $z=z_0$ which is a limiting point of zeros of $f(z)$ must be an isolated essential singularity.

4.64. Limiting Point of Poles. Let $z=z_0$ be a limiting point of a sequence of the poles of the one-valued function $f(z)$, then in every neighbourhood of z_0, there are poles of the given function. The point $z=z_0$ is, therefore, a singlarity of $f(z)$. It cannot a be pole, since it is not isolated (§ 4.62). Such a singularity is termed a **non-isolated essential singularity**, or simply an **essential singularity**.

4.7. Casorati's Theorem.* If z_0 is an isolated essential singular point of $f(z)_0$ and β is any finite number, then there is a point z in the circle $|z-z_0| < r$ at which $|f(z)-\beta| < \varepsilon$, where r and ε are arbitrarily small positive numbers.

* Commonly attributed to Weierstrass.

Let $f(z)$ have the Laurent expansion

$$f(z)=\sum_{0}^{\infty} a_n(z-z_\circ)^n+\sum_{1}^{\infty} b_n\,(z-z_o)^{-n}$$

in the annulus $0<\ |\ z-z_\circ\ |<r$.

It can be easily shown that there are values of z in the circle $|\ z-z_\circ\ |<r$ at which $|\ f(z)\ |\ >M$, where M is any positive number.

For, suppose that $|\ f(z)\ |\leqslant M$ on C, the circle $|\ z-z_\circ\ |=\rho<r$,

then $$|\ b_n\ |=\left|\ \frac{1}{2\pi i}\int_C (z-z_\circ)^{n-1} f(z)\,dz\ \right|\leqslant M\rho^n,$$ by § 4.13.

This is true for all positive values of n and ρ and as $\rho\to 0$, all the coefficients b_n would be zero which means that $z_\circ$ is not an isolated essential singularity.

Next, if the equation $f(z)=\beta$ has roots within the circle $|\ z-z_\circ\ |=r$, however small r may be, the theorem is proved. If $f(z)=\beta$ does not have an infinite number of roots in the neighbourhood of the point $z_\circ$, we can choose r so small that $f(z)-\beta$ has no zero for $|\ z-z_\circ\ |<r$. The function $\phi(z)=\dfrac{1}{f(z)-\beta}$ is then analytic for $|\ z-z_\circ\ |<r$ except at $z=z_\circ$. The point $z=z_\circ$ cannot be anything but an isolated essential singularity for $\phi(z)$; for, since $f(z)=\dfrac{1}{\phi(z)}+\beta$, $f(z)$ would be analytic if $\phi(z)$ had a pole, while $f(z)$ would be analytic or have a pole if $\phi(z)$ were analytic.

From what we have just proved in the beginning, it follows that there is a point z in $|\ z-z_\circ\ |<r$ such that $|\ \phi(z)\ |>\dfrac{1}{\varepsilon}$,

i. e., $$|\ f(z)-\beta\ |<\varepsilon,$$

however small the positive number ε may be.

Note. This theorem sharply distinguishes between poles and isolated essential singularities. While the absolute value of the function tends to infinity in the neighbourhood of a pole, the value of the function at an isolated essential singularity is indeterminate and in fact differs by an arbitrarily small number from any preassigned value an infinity of times.

A more precise result is Picard's theorem, which states that, in every neighbourhood of an isolated essential singularity, there exists a point at which the function actually attains any given value with at most one exception.

4.8. The Point at Infinity. By writing $z=\frac{1}{z'}$, 'infinity' may be regarded as a point corresponding to the value $z'=0$. The behaviour of a function $f(z)$ at 'infinity' is then determined by the behaviour of $f\left(\frac{1}{z'}\right)$ at $z'=0$. Accordingly, we shall say that $f(z)$ is analytic, or has a simple pole, or has an isolated essential singularity, if $f\left(\frac{1}{z'}\right)$ has the same property at $z'=0$.

Thus, $f(z)$ has a zero of order m at infinity, if $\phi(z')=f\left(\frac{1}{z'}\right)$ has a zero of order m at $z'=0$,

i. e., if $$\phi(z')=\sum_{n=m}^{\infty} a_n z'^n ;$$

$$\therefore \quad f(z)=\sum_{n=m}^{\infty} \frac{a_n}{z^n}=\frac{a_m}{z_m}+\frac{a_{m+1}}{z^{m+1}}+\dots . \qquad \dots\dots\dots(\mathrm{i})$$

Since, least value of $m=1$, it follows that if $f(z)$ has a zero of order m at infinity, then its expansion does not contain the constant term and positive powers of z, i. e., it can be expanded only in negative powers of z with the constant term missing.

Again, $f(z)$ has a pole of order m at infinity, if $\phi(z')=f\left(\frac{1}{z'}\right)$ has a pole of order m at $z'=0$,

i. e., if $\phi(z')=\sum_{0}^{\infty} a_n z'^n+\sum_{1}^{m} b_n z'^{-n}$;

$$\therefore \quad f(z)=\sum_{n=1}^{m} b_n z^n+\sum_{0}^{\infty} a_n z^{-n}. \qquad \dots\dots\dots(2)$$

Thus, $f(z)$ has a pole of order m at $z=\infty$, if the principal part of $\phi(z')$ at $z'=0$, i. e., the principal part of $f(z)$ at $z=\infty$ is a finite series; in other words, the series in positive powers of z in the expansion of $f(z)$ is a finite series.

Since a polynomial $a_0+a_1z+a_2z^2+\ldots a_mz^m$ is analytic at every finite point, it may be regarded as the principal part of a Laurent series about $z=\infty$. In other words, $z=\infty$ is a pole of order m of every polynomial of the mth degree.

4.81. If f(z) is analytic in the whole z-plane including the point at infinity, then f(z) is a constant.

Since $f(z)$ is analytic for all finite values of z, it has the Taylor expansion $$f(z)=\sum_0^\infty a_nz^n,$$

whence $$\phi(z')=f\left(\frac{1}{z'}\right)=\sum_0^\infty a_nz'^{-n}.$$

Now, $\phi(z')$ is analytic at $z'=0$; $\therefore$ $a_1=a_2=\ldots a_n=\ldots=0$.

Hence $f(z)=a_0$, a constant.

Note. The whole z-plane including the point at infinity is called the *extended* z-plane.

4.82. Rational Functions.

Theorem. *A single-valued function which has no singularities other than poles at any point (including the point at infinity) is a rational function. Conversely, a rational function has no singularities other than poles.*

Let $f(z)$ be such a function, then $f(z)$ can have only a finite number of poles; for, otherwise the poles would have a limiting point, either at a finite point or at infinity which would be a non-isolated essential singularity of $f(z)$, contrary to hypothesis.

It follows that if at $z_1, z_2, \ldots, z_k$, points in a finite part of the z-plane, $f(z)$ has poles of orders $n_1, n_2, \ldots, n_k$ respectively, then

$$\phi(z)=(z-z_1)^{n_1}(z-z_2)^{n_2}\ldots(z-z_k)^{n_k}f(z)$$

is analytic for all finite values of z. Hence, for finite values of z, $\phi(z)$ has the Maclaurin expansion $$\phi(z)=\sum_0^\infty a_nz^n.$$

Since $\phi(z)$ can have a pole at infinity, the series for $\phi(z)$ must terminate [see §4·8], *i.e.*, $\phi(z)$ is a polynomial.

Thus, $f(z)=\dfrac{\phi(z)}{(z-z_1)^{n_1}(z-z_2)^{n_2}\ldots(z-z_k)^{n_k}}$, a quotient of two polynomials, is a rational function.

Again, a rational function $f(z)=\dfrac{P(z)}{Q(z)}$,

where $P(z)=a_0+a_1z+\ldots+a_nz^n$, and $Q(z)=b_0+b_1z+\ldots+b_mz^m$, can have singularities only at the zeros of the denominator. We may suppose that $P(z)$ and $Q(z)$ have no common zeros ; for, if z_1 is a zero of order p of $P(z)$ and also a zero of order k of $Q(z)$, then $P(z)=(z-z_1)^p\,P_1(z)$, $P_1(z_1)\neq0$ and $Q(z)=(z-z_1)^k\,Q_1(z)$, $Q_1(z_1)\neq0$, since $P(z)$ and $Q(z)$ are analytic functions for all finite values of z. Thus, in writing $f(z)$ as $\dfrac{P(z)}{Q(z)}$, common factors can be removed so that $f(z)$ can be written as the quotient of two polynomials with no common zeros. It follows, therefore, that $\dfrac{1}{f(z)}$ is analytic at the zeros of $Q(z)$ and at these zeros of $Q(z)$, $f(z)$ cannot have singularities other than poles [see §4·6 III].

Hence, a rational function has no singularities other than poles.

Ex. 1. *What kind of singularity has the function.*

(i) e^z at $z=\infty$;

(ii) $\sin\dfrac{1}{z}$ at $z=0$; *(Agra 1944)*

(iii) $\cot z$ at $z=\infty$.

(iv) $\sec\dfrac{1}{z}$ at $z=0$.

(i) Let $z=\dfrac{1}{z'}$, then $e^z=e^{1/z'}=\sum_0^\infty \dfrac{1}{z'^n\lfloor n}$

has an isolated essential singularity at $z'=0$, *i.e.*, at $z=\infty$.

(ii) The zeros of $\sin\dfrac{1}{z}$ are $\dfrac{1}{n\pi}$ $(n=\pm1,\ \pm2,\ \pm3,\ldots)$ and $z=0$

is their limit point. Hence, $z=0$ is an isolated essential singularity of $\sin \frac{1}{z}$.

(iii) The function has simple poles at $n\pi$ $(n=0, \pm1, \pm2,\ldots)$ and $z=\infty$ is their limit. Hence, $z=\infty$ is a non-isolated essential singularity of $\cot z$.

(iv) Here $\cos \frac{1}{z}=0$, when $\frac{1}{z}=\frac{2n+1}{2}\pi$ $(n=0, \pm1, \pm2,\ldots)$,

$$i.e.,\ z=\frac{1}{(n+\frac{1}{2})\pi}\ ;$$

$\therefore$ $z=0$ is the limit point of the poles of $\sec \frac{1}{z}$.

Hence, $\sec \frac{1}{z}$ has a non-isolated essential singularity at $z=0$.

Note on (i). In §4·7, it has been stated that in every neighbourhood of an isolated essential singularity, the function actually attains any arbitrary value with at most one exception. This example shows that such an exception can occur, since e^z can never take the value 0 near $z=\infty$.

Ex. 2. *Show that the function* $e^{\frac{1}{z}}$ *actually takes every value except zero an infinity of times in the neighbourhood of* $z=0$.

The function $e^{\frac{1}{z}}$ is analytic for all values of z except $z=0$;

$\therefore$ the Laurent expansion is $e^{\frac{1}{z}}=\sum_0^\infty \frac{1}{z^n \lfloor n}$.

The principal part is an infinite series and, therefore, $z=0$ is an isolated essential singularity. Hence, by §4·7, the result follows.

Aliter. Let us assign the value A, $A\neq0$, to $e^{\frac{1}{z}}$, then the values of z for which $e^{\frac{1}{z}}=A$ are given by $\frac{1}{z}=\log r+i\,(0+2k\pi.)$,

where $A=re^{i0}$ and k is any integer, positive or negative.

We shall have $|z|<\rho$, however small ρ may be,

if $(\log r)^2+(0+2k\pi)^2>\frac{1}{\rho^2}$, which condition is satisfied by an infinity of values of the integer k.

Note. $e^{\frac{1}{z}} \to 0$, if $z \to 0$ along the negative real axis.

Ex. 3. *Find the singularities of the function* $\dfrac{e^{c/(z-a)}}{e^{z/a}-1}$, *indicating the character of each singularity.* *(Agra 1914)*

The function $f(z)=\dfrac{e^{c/(z-a)}}{e^{z/a}-1}$ is analytic for all values of z other than $z=a$, $z=0$, and $z=2n\pi ia$ $(n=\pm 1, \pm 2, \ldots)$.

At $z=0$, the denominator has a simple zero while the numerator is analytic. Hence, the function has a simple pole at $z=0$, by § 4·6. Similarly, there is a simple pole at each of the points $2n\pi ia$ $(n=\pm 1, \pm 2, \ldots)$.

At $z=a$, the numerator has an isolated singularity and the function has a Laurent expression which may be obtained as follows :

$$f(z)=\frac{e^{c/(z-a)}}{e^{1+(z-a)/a}-1}=\frac{1+\dfrac{c}{z-a}+\left(\dfrac{c}{z-a}\right)^2\cdot\dfrac{1}{\lfloor 2}+\cdots}{e\left[1+\dfrac{z-a}{a}+\left(\dfrac{z-a}{a}\right)^2\dfrac{1}{\lfloor 2}+\ldots\right]-1}.$$

The expression involves all positive and negative powers of $(z-a)$; hence, the function has an isolated essential singularity at $z=a$.

Ex. 4. *Consider the singularities of the function represented by the series* $\sum_{n=0}^{\infty}\dfrac{1}{\lfloor n}\cdot\dfrac{1}{1+(2^n z)^2}$, *and obtain its expansion by Laurent's theorem.* *(Agra 1932)*

The denominator has simple zeros at $z=-\dfrac{i}{2^n}$ and $z=\dfrac{-i}{2^n}$ $(n=0, 1, 2, \ldots)$. Hence by §4·6, the function has simple poles at each of these points. All these singularities lie within the circle $|z|=1$ and have $z=0$ as a limit point which is, therefore, a non-isolated essential singularity of the function (§ 4·64).

The function is analytic on and outside a circle C, $|z|=k(k>1)$, and therefore, has the Laurent expanssion $\dfrac{b_2}{z^2}+\dfrac{b_4}{z^4}+\dfrac{b_6}{z^6}+\ldots$, where $b_{2m+1}=0$, since the function is an

even function of z, and $b_{2m}=\frac{1}{2\pi i}\int_C z^{2m-1}\left(\sum_{n=0}^{\infty}\frac{1}{\lfloor n}\frac{2^{-2n}}{2^{-2n}+z^2}\right)dz$.

Now, $z^{2m-1}\sum_{n=0}^{\infty}\frac{1}{\lfloor n}\frac{2^{-2n}}{2^{-2n}+z^2}=z^{2m-3}\sum_{n=0}^{\infty}\frac{2^{-2n}}{\lfloor n}\left(1+2^{-2n}\ z^{-2}\right)^{-1}$

$$=z^{2m-3}\left[\left(1-z^{-2}+z^{-4}-\ldots\right)+\frac{2^{-2}}{\lfloor 1}\left(1-2^{-2}\ z^{-2}+\ldots\right)-\ldots\ldots\right]$$

$$=z^{2m-3}\left[\left(1+\frac{2^{-2}}{\lfloor 1}+\frac{2^{-4}}{\lfloor 2}+\frac{2^{-6}}{\lfloor 3}+\ldots\right)\right.$$

$$-z^{-2}\left(1+\frac{2^{-4}}{\lfloor 1}+\frac{2^{-8}}{\lfloor 2}+\ldots\right)+\ldots$$

$$\left.+(-1)^{m-1}\ (z^{-2})^{m-1}\left(1+\frac{2^{-2m}}{\lfloor 1}+\ldots\right)+\ldots\ldots\right]$$

$$=z^{2m-3}\ \ e^{2^{-2}}-z^{2m-5}\ \ e^{2^{-4}}+\ldots+(-1)^{m-1}\ \ z^{-1}\ e^{2^{-2m}}+\ldots\ .$$

The only term in the expression which furnishes a non-zero integral is the term in z^{-1} ; for, putting $z=ke^{i0}$ $(k>1)$, any other term in the integrand will contain powers of $e^{\pm i\theta}$ and the integral between the limits 0 and 2π of such a term will be zero.

Since the coefficient of $z^{-1}=(-1)^{m-1}\ e^{2^{-2m}}$, we have

$$b_{2m}=\ \frac{1}{2\pi i}\int_C(-1)^{m-1}\ e^{2^{-2m}}\frac{dz}{z}$$

$$=\ \frac{(-1)^{m-1}}{2\pi i}\ \ e^{2^{-2m}}\int_0^{2\pi}id0,\ \text{where}\ z=ke^{i\theta}\ .$$

$$=\ (-1)^{m-1}\ \ e^{2^{-2m}}.$$

Therefore, when $|\ z\ |>1$, the function can be expanded in the form

$$\frac{e^{2^{-2}}}{z^2}-\frac{e^{2^{-4}}}{z^4}+\frac{e^{2^{-6}}}{z^6}-\ldots\ldots.$$

By putting $z=\frac{1}{z'}$, we see that the function in z' has a zero of second order at $z'=0$; hence, the given function has a zero of second order at infinity.

Ex. 5. *The only singularities of a single-valued function $f(z)$ are poles of order 1 and 2 at $z=-1$ and $z=2$, with residues at these poles 1 and 2 respectively. If $f(0)=\frac{7}{4}$, $f(1)=\frac{5}{2}$, determine the function and expand it in a Laurent series valid in $1<|z|<2$.*

Since the only singularities of $f(z)$ are poles of order 1 and 2 at $z=-1$ and $z=2$ respectively, $f(z)$ is a rational function of the form [see § 4·82]

$$f(z)=a_0+a_1z+\ldots+a_mz^m+\frac{1}{z+1}+\frac{2}{z-2}+\frac{b}{(z-2)^2}$$

[$\because$ residues at -1 and 2 are respectively 1 and 2.]

Further, since $z=\infty$ is not a pole, $a_1=a_2=\ldots=a_m=0$ [see § 4·8].

Hence, $f(z)=a_0+\frac{1}{z+1}+\frac{2}{z-2}+\frac{b}{(z-2)^2}$.

Now, $\frac{7}{4}=f(0)=a_0+\frac{b}{4}$, and $\frac{5}{2}=f(1)=a_0-\frac{3}{2}+b$;

$\therefore \quad a_0=1$ and $b=3$.

Thus, the required function is

$$f(z)=1+\frac{1}{z+1}+\frac{2}{z-2}+\frac{3}{(z-2)^2}=\frac{z^3-3z^2+7}{(z+1)(z-2)^2}.$$

The Laurent expansion valid in $1<|z|<2$ is given by,

$$f(z)=1+\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}-1\left(1-\frac{z}{2}\right)^{-1}+\frac{3}{4}\left(1-\frac{z}{2}\right)^{-2}$$

$$=1+\frac{1}{z}\sum_0^\infty(-1)^n/z^n-\sum_0^\infty\left(\frac{z}{2}\right)^n+\frac{3}{4}\sum_0^\infty(n+1)\left(\frac{z}{2}\right)^n.$$

$$=1+\sum_0^\infty\left\{\frac{3}{4}(n+1)-1\right\}\left(\frac{z}{2}\right)^n+\sum_0^\infty\frac{(-1)^n}{z^{n+1}}$$

$$=1+\sum_0^\infty\frac{3n-1}{3}\left(\frac{z}{2}\right)^n+\sum_0^\infty\frac{(-1)^n}{z^{n+1}}\ .$$

Exercise 4

1. Show that the integral of $\frac{1}{z}$ along a semi-circular arc from -1 to $+1$ has the vaule $-\pi i$ or πi according as the arc lies above or below the real axis.

2. If the function $f(z)$ is analytic in $|z-a|<R$, prove that for $o<r<R$, $f'(a)=\frac{1}{\pi r}\int_0^{2\pi} P(0)e^{-i0}\,d0$,

where $P(0)$ is the real part of $f(a+re^{i0})$.

(Agra 1957, Vikram 1959

3. By using the integral representation of $f^n(z_0)$, prove that

$$\left(\frac{a^n}{\lfloor n}\right)^2=\frac{1}{2\pi i}\int_C \frac{a^n e^{az}}{\lfloor n\, z^{n+1}}\,dz,$$

where C is any closed contour surrounding the origin. Hence, prove that

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{\lfloor n}\right)^2=\frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos 0}\,d0 \qquad (Agra\ 1956)$$

4. Let $f(z)$ be analytic in the unit circle and let $f(0)=1$; by evaluating the integrals $\frac{1}{2\pi}\int_{|z|=1}\left[2\pm(z+\frac{1}{z})\right]f(z)\frac{dz}{z}$ and by introducing polar coordinates, show that

$$\frac{2}{\pi}\int_0^{2\pi} f(e^{i0})\cos^2\frac{0}{2}\,d0=2+f'(0),\ \frac{2}{\pi}\int_0^{2\pi} f(e^{i0})\sin^2\frac{0}{2}\,d0=2-f'(0).$$

Use this result in order to prove the following statement : If $f(z)$ is analytic in $|z|\leqslant 1, f(0)=1$ and $\mathrm{Re}\{f(z)\}\geqslant$ in $|z|\leqslant 1$, then $-2\leqslant \mathrm{Re}\{f'(0)\}\leqslant 2$.

5. If $f(z)$ is analytic and $|f(z)|\leqslant 1$ in $|z|\leqslant 1$, prove that $|f'(0)|\leqslant 1$.

6. The Legendre polynomial $P_n(z)$ is defined by

$$P_n(z)=\frac{1}{2^n\lfloor n}\frac{d^n}{dz^n}\left[(z^2-1)^n\right];$$

show that $P_n(z)$ can be represented by the contour integral

$$P_n(z)=\frac{1}{2\pi i}\int_C \frac{(\zeta^2-1)^n}{2^n(\zeta-z)^{n+1}}\,d\zeta,$$

where C surrounds the point z. Taking, in particular, C to be the circle of centre z and radius $\sqrt{|z^2-1|}$, deduce Laplace's formula

$$P_n(z)=\frac{1}{\pi}\int_0^{\pi}(z+\sqrt{z^2-1}\cos\theta)^n d\theta.$$

7. Find the function $f(z)$, which is analytic within the circle $|z|=1$ and has the value $\dfrac{a-\cos\theta+i\sin\theta}{a^2-2a\cos\theta+1}$ (where $a>1$ and θ is the vectorial angle) at points on the circumference of the circle $|z|=1$. *(Agra 1930)*

8. Determine a function which shall be analytic within the circle $|z|=1$ and shall have at the circumference of this circle the value $\dfrac{(a^2-1)\cos\theta+i(a^2+1)\sin\theta}{a^4-2a^2\cos 2\theta+1}$,

where $a^2>1$, θ being the vectorial angle. *(Vikram 1960)*

9. Find the different developments of $\dfrac{1}{(z-1)(z-3)}$ is powers of z, according to the position of the point z in the plane. Expand this function in a Taylor's series about $z=2$, and indicate the circle of convergence. *(Agra 1943)*

10. Establish the expansion

$$f(z)=f(a)+2\left\{\frac{z+a}{2}f'\left(\frac{z-a}{2}\right)+\frac{(z+a)^3}{2^3\lfloor 3}+f'''\left(\frac{z+a}{2}\right)+\frac{(z-a)^5}{2^5\lfloor 5}f^{(5)}\left(\frac{z+a}{2}\right)+\cdots\cdots\right\},$$

and show that the expansion is valid as long as z lies within the neighbourhood of the point $\frac{1}{2}(z+a)$.

11. Find the Taylor or Laurent series which represent the function

$$\frac{1}{(z^2+1)(z+2)},$$

(i) when $|z|<1$,
(ii) when $1<|z|<2$,
(iii) when $|z|>2$.

12. If $f(z)=\sum_{1}^{\infty}\frac{z^2}{n^2z^2+a^2}$, show that $f(z)$ is finite and continuous for all real values of z, but cannot be expanded as a Maclaurin's series in ascending powers of z. Show that $f(z)$ possesses Laurent expansions valid in a succession of rings.

13. Show that $\cosh\left(z+\frac{1}{z}\right)=a_0+\sum_{1}^{\infty}a_n\left(z^n+\frac{1}{z^n}\right)$,

where $a_n=\frac{1}{2\pi}\int_0^{2\pi}\cos n\theta\cosh(2\cos\theta)\,d\theta$. (*Agra 1945*)

14. Show that

$$e^{\frac{1}{2}c(z-\frac{1}{z})}=\sum_{n=-\infty}^{\infty}a_nz^n,$$

where $a_n=\frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta-c\sin\theta)\,d\theta$. (*Agra 1938, '52, '60*)

15. Show that

$$e^{cz+d/z}=a_0+a_1z+a_2z^2+\ldots+\frac{b_1}{z}+\frac{b_2}{z^2}+\ldots,$$

where $a_n=\frac{1}{2\pi}\int_0^{2\pi}e^{(c+d)\cos\theta}\cos\{(c-d)\sin\theta-n\theta\}d\theta,$

and $b_n=\frac{1}{2\pi}\int_0^{2\pi}e^{(c+d)\cos\theta}\cos\{(d-c)\sin\theta-n\theta\}d\theta.$

16. Show that in the annulus defined by $|a|<|z|<|b|$, the function $\left\{\frac{bz}{(z-a)(b-z)}\right\}^{\frac{1}{2}}$ can be expanded in the form

$$S_0+\sum_{n=1}^{\infty}S_n\left(\frac{a^n}{z^n}+\frac{z^n}{b^n}\right),$$

where $S_n=\sum_{l=0}^{\infty}\frac{1.3\ldots(2l-1)1.3\ldots(2l+2n-1)}{2^{2l+n}\lfloor l\rfloor l+n}\left(\frac{a}{b}\right)^l$

17. Prove that if L is a positive constont, and $(1-2zh+h^2)^{-\frac{1}{2}}$ is expanded in $1+h\,P_1(z)+h^2\,P_2(z)+h^3\,P_3(z)+\ldots\ldots$, where $P_n(z)$ is a polynomial of degree n in z, then this series converges so long as is z in the interior of an ellipse whose foci are points $z=1$ and $z=-1$ and whose semi-major axis is $\frac{1}{2}(h+h^{-1})$.

(*Agra 1932*)

18. Prove that, when $|z|<1$, the principal value of $(1+z)^a$ is equal to

$$1+\sum_{n=1}^{\infty}\frac{a\,(a-1)\ldots(a-n+1)}{\lfloor n}\,z^n.$$

19. The function $f(z)$ has a simple pole at $z=a$. Show that the residue at this pole is $\operatorname{Lt}_{z\to a}\{(z-a)\,f(z)\}$.

20. Applying Parseval's identity (Ex. 2, §4·62) to the function

$$\frac{1}{(1-z)^2}=\sum_{n=1}^{\infty}nz^{n-1},\ |z|<1,$$

show that

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(1-2\rho\cos\theta+\rho^2)^2}=\sum_{n=1}^{\infty}n^2\rho^{2n-2}=\frac{1+\rho^2}{(1-\rho^2)^3},\quad 0\leqslant\rho<1.$$

21. Applying Parseval's identity to the function

$$1+z+z^2+\ldots+z^{n-1}=\frac{1-z^n}{1-z},$$

deduce the formula

$$\int_0^{2\pi}\left(\frac{\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}\right)^2 d\theta=2\pi n.$$

22. What kind of singularity has the function

(i) $(z^2+4)/e^z$ at $z=\infty$;

(ii) $\cos z-\sin z$ at $z=\infty$;

(iii) $\dfrac{1}{\cos z}$ at $z=\infty$;

(iv) $z\operatorname{cosec} z$ at $z=\infty$,

(v) e^{-1/z^2} at $z=\infty$;

(vi) $\sin\dfrac{1}{1-z}$ at $z=\infty$, and $z=1$:

(vii) $\dfrac{1-e^z}{1+e^z}$ at $z=\infty$;

(viii) $\dfrac{\cot\pi z}{(z-a)^2}$ at $z=a$;

(ix) $\dfrac{1}{1-e^z}$ at $z=2\pi i$;

(x) $\dfrac{1}{\sin z-\cos z}$ at $z=\dfrac{\pi}{4}$;

(xi) $\operatorname{cosec}\dfrac{1}{z}$ at $z=0$;

(xii) $\tan\dfrac{1}{z}$ at $z=0$.

23. Classify the points $z=0$, $z=1$ and the point at ∞, in relation to the function $f(z)=\dfrac{z-2}{z^2}\sin\dfrac{1}{1-z}$.

24. What essential difference is there between of the behaviour of the real function $y=\begin{cases} e^{-1/x^2}, & x\neq 0, \\ 0, & x=0 ; \end{cases}$

and that of the function of a complex variable $w=e^{-1/z^2}$ in a neighbourhood of the origin.

25. The only singularities of a single-valued function $f(z)$ are poles of order 2 and 1 at $z=1$ and $z=2$, with residues of these poles 1 and 3 respectively. If $f(0)=\dfrac{3}{2}$, $f(-1)=1$, determine the function.

ANSWERS

7. $\dfrac{1}{a-z}$. 8. $\dfrac{z}{z^2-a^2}$.

9. (*i*) $\sum_0^\infty\left\{\dfrac{1}{2}\left(1-\dfrac{1}{3^{n+1}}\right)z^n\right\}$ for $|z|<1$;

(*ii*) $-\dfrac{1}{6}\sum_0^\infty\left(\dfrac{z}{3}\right)^n-\dfrac{1}{2}\sum_0^\infty\dfrac{1}{z^{n+1}}$ for $1<|z|<3$;

(*iii*) $\dfrac{1}{2}\sum_0^\infty\dfrac{3^n-1}{z^{n+1}}$ for $|z|>3$;

(*iv*) $-\sum_0^\infty (z-2)^{2n}$ for $|z-2|<1$.

11. (*i*) $\frac{1}{10}\sum_0^\infty (-1)^n \left(\frac{z}{2}\right)^n + \frac{1}{5}(2-z)\sum_0^\infty (-1)^n z^{2n}$;

(*ii*) $\frac{1}{10}\sum_0^\infty (-1)^n \left(\frac{z}{2}\right)^n + \frac{1}{5}\left(\frac{2}{z^2}-\frac{1}{z}\right)\sum_0^\infty (-1)^n \left(\frac{1}{z^2}\right)^n$;

(*iii*) $\frac{1}{5}\sum_0^\infty (-1)^n \left(\frac{2}{z}\right)^n + \frac{1}{5}\left(\frac{2}{z^2}-\frac{1}{z}\right)\sum_0^\infty (-1)^n \frac{1}{z^{2n}}$.

22. (*i*) and (*ii*) Isolated essential singularity ;

(*iii*), (*iv*), (*vii*), (*xi*) and (*xii*) Non-isolated essential singularity ;

(*v*) No sigularity ;

(*vi*) No singularity at $z=\infty$ which is a simple zero, isolated essential singularity at $z=1$;

(*viii*) A double pole, if a is not an integer or zero and a triple pole, if a is an integer ;

(*ix*) & (*x*) Simple pole.

23. A double pole at $z=0$, an isolated essential singularity at $z=1$ and a double zero at $z=\infty$.

25. $\frac{2z^3-4z^2+3z-3}{(z-1)^2(z-2)}$.

CHAPTER V

THE CALCULUS OF RESIDUES

51. The Residue at a Singularity. We have already shown (§ 4·6) that in the neighbourhood of an isolated singularity $z=z_o$, a single-valued function $f(z)$ has the Laurent expansion

$$f(z)=\sum_{0}^{\infty} a_n (z-z_o)^n+\sum_{0}^{\infty} b_n (z-z_o)^{-n}.$$

The coefficient b_1 [defined to be the residue of $f(z)$ at z_o]

$=-\frac{1}{2\pi i}\int_{C'} f(z)dz$,.....(1) where C' is any circle $|z-z_o|=r$ such that $f(z)$ has no other singularity within it. The value of the integral in (1), however, is not changed if we replace C' by a closed contour C (large or small) surrounding the point z_o such that $f(z)$ is analytic in C, except at $z=z_o$; for, r can be taken so small that C' is interior to C and thus $f(z)$ is analytic in the annulus between C' and C. Here, by Cauchy's theorem, we have

$$\int_C f(z)\,dz=\int_{C'} f(z)\,dz,$$

$$i.e.,\quad b_1=\frac{1}{2\pi i}\int_C f(z)\,dz. \qquad \text{.....(2)}$$

If the pole at $z=z_o$ be of order m, then

$\phi(z)=(z-z_o)^m f(z)$ is analytic at $z=z_o$,

and $\phi(z_o)=\underset{z\to z_o}{\text{Lt}}\ \{(z-z_o)^m\ f(z)\}=b_m\neq 0$; hence, $\phi(z)$ has the Taylor expansion

$$\phi(z)=\sum_{n=0}^{\infty} a_n (z-z_o)^{m+n}+\sum_{n=1}^{m} b_n (z-z_o)^{m-n}$$

$$=\sum_{0}^{\infty} (z-z_o)^n \frac{\phi^n(z_o)}{\lfloor n}.$$

Therefore, b_1 = coefficient of $(z-z_\circ)^{m-1} = \dfrac{\phi^{m-1}(z_\circ)}{\lfloor m-1}$(3)

When $m=1$, this formula becomes

$$b_1 = \phi(z_\circ) = \underset{z \to z_o}{\mathrm{Lt}} \; (z-z_\circ) f(z). \qquad \text{.....(4)}$$

It follows that, if $P(z)$ and $Q(z)$ are both analytic at $z=z_o$ and if $f(z) = \dfrac{P(z)}{Q(z)}$ has a simple pole at $z=z_o$, then the residue of $f(z)$ at $z=z_0$ is $\dfrac{P(z)}{Q'(z_0)}$, where $P(z_0) \neq 0$;

for, $$b_1 = \underset{z \to z_o}{\mathrm{Lt}} \; (z-z_o) f(z)$$

$$= \underset{z \to z_o}{\mathrm{Lt}} \; \frac{(z-z_o)\,[P(z_o)+P'(z_o)\,(z-z_o)+\ldots]}{Q'(z_o)\,(z-z_o)+\dfrac{Q''(z_o)}{\lfloor 2}\,(z-z_o)^2+\ldots} ,$$

$Q(z_o)=0$, since z_o is a simple pole of $f(z)$, *i.e.*, a simple zero of $Q(z)$;

$$\therefore \quad b_1 = \frac{P(z_o)}{Q'(z_o)}. \qquad \text{.....(5)}$$

When $z=z_o$ is a pole of order m of the function $f(z)=\dfrac{P(z)}{Q(z)}$ and $P(z_o) \neq 0$, the residue at z_o can be calculated similarly.

For example, when $m=2$,

$$\frac{P(z)}{Q(z)} = \frac{P(z_o)+P'(z_o)\,(z-z_o)+P''(z_o)\,(z-z_o)^2/\lfloor 2 +\ldots}{Q''(z_o)\dfrac{(z-z_o)^2}{\lfloor 2}+\; Q'''(z_o)\;\dfrac{(z-z_o)_3}{\lfloor 3}+\ldots}$$

i.e.,
$$\frac{P(z)}{Q(z)} = \frac{1}{\dfrac{(z-z_o)^2}{\lfloor 2}Q'''(z_o)}\left\{P(z_o)+P'(z_o)\,(z-z_o)\right.$$

$$\left.+P''(z_o)\frac{(z-z_o)^2}{\lfloor 2}+\ldots\right\}\left\{1+\frac{z-z_o}{3}\;\frac{Q''''(z)}{Q'''(z)}+\ldots\right\}^{-1}$$

$$= \frac{2}{(z-z_0)^2 Q''(z_0)} \left\{ P(z_0) + P'(z_0)\,(z-z_0) + \cdots \right\}$$

$$\times \left\{ 1 - \frac{z-z_0}{3} \cdot \frac{Q'''(z_0)}{Q''(z_0)} + \cdots \right\};$$

$\therefore$ residue at z_0 = coefficient of $\dfrac{1}{z-z_0}$ in the above expansion.

$$= \frac{2}{Q''(z_0)} \left[P'(z_0) - \tfrac{1}{3} P(z_0) \cdot \frac{Q'''(z_0)}{Q''(z_0)} \right]$$

$$= \frac{6P'(z_0)Q''(z_0) - 2P(z_0)Q'''(z_0)}{3\{Q''(z_0)\}^2}. \qquad \ldots\ldots\ldots(6)$$

Note. *At a multiple pole $z=z_0$ of $f(z)$, the residue can be easily determined in some cases by putting $z=z_0+t$ and expanding $f(z_0+t)$ in ascending powers of t, where $|t|$ is sufficiently small. The residue is the coefficient of $\dfrac{1}{t}$ in this expansion.*

Ex. 1. *Find the residues of :*

(*i*) $e^{az}/(1+e^z)$ *at* $z=\pi i$. *(Agra 1937)*

(*ii*) $\dfrac{1}{(z^2+a^2)^2}$ *at* $z=ai$.

(i) $e^z+1=0$, when $e^z=-1=e^{(2n+1)\pi i}$ $(n=0, \pm 1, +2, \ldots)$;

$\therefore$ $z=\pi i$ is a simple zero of $1+e^z$, *i.e.*, a simple pole of $\dfrac{e^{az}}{1+e^z}$.

Hence, residue at $\pi i = \left[\dfrac{e^{az}}{\dfrac{d}{dz}(1+e^z)} \right]_{z=\pi i}$, by §5·1(5).

$$= \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}.$$

(ii) Here $z=ai$ is a double pole of the given function.

Let $\phi(z) = (z-ai)^2 \dfrac{1}{(z^2+a^2)^2} = \dfrac{1}{(z+ai)^2}$,

then, by §5.1 (3), the required residue $=\phi'(ai)=\dfrac{-2}{(ai+ai)^3}$

$$=\frac{1}{4a^3i}.$$

Another Method. Putting $z=ai+t$, we have

$$\frac{1}{(z^2+a^2)^2}=\frac{1}{t^2(2ai+t)^2}=-\frac{1}{4a^2t^2}\left(1+\frac{t}{2ai}\right)^{-2}.$$

When $|t|$ is sufficiently small, the coefficient of $\frac{1}{t}$ in the expansion is $-\dfrac{1}{4a^2}\cdot\dfrac{-2}{2ai}=\dfrac{1}{4a^3i}$, which is, therefore, the required residue.

5. 2. Cauchy's Residue Theorem. *If $f(z)$ be analytic inside and on a closed contour C, except at a finite number of poles $z_1, z_2, \ldots, z_n$ within C, then* $\displaystyle\int_C f(z)dz=2\pi i\sum_{k=1}^{n} R_{z_k}$,

where R_{z_k} denotes the residue of $f(z)$ at z_k.

Let $C_1, C_2, \ldots, C_n$ be circles with centres at $z_1, z_2, \ldots, z_n$ respectively and radii so small that they lie inside C and do not overlap (see fig. 27). These circles together with the curve form the boundary of a multiply-connected region in which $f(z)$ is analytic, so that, by Cauchy's theorem (§4.24)

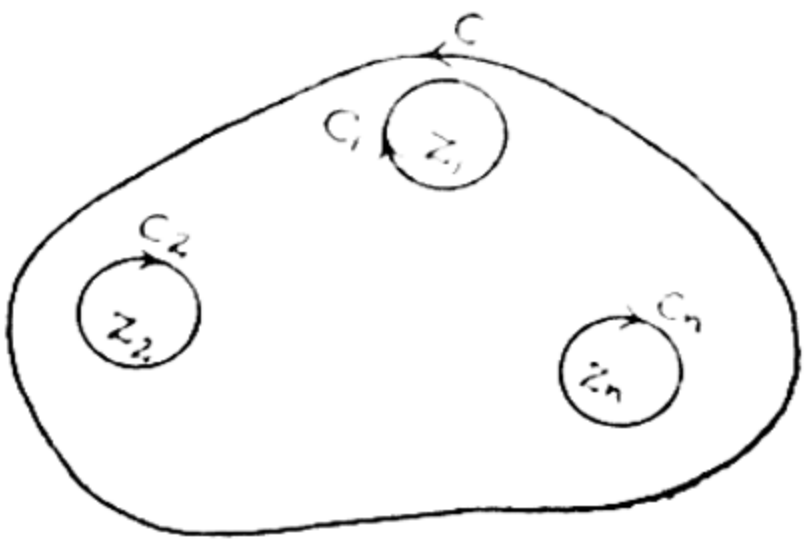


Fig. 27

$$\int_C f(z)dz=\sum_{k=1}^{n}\int_{C_k} f(z)\,dz.$$

But $\displaystyle\int_{C_k} f(z)dz=2\pi iR_{z_k}$ (§5 1).

Hence, $\displaystyle\int_C f(z)dz=2\pi i\sum_{k=1}^{n} R_{z_k}$.

5.21. Residue at Infinity. If $f(z)$ has a pole at infinity, then the residue of $f(z)$ at infinity is defined to be $-\frac{1}{2\pi i}\int_C f(z)\,dz$, where C is a closed contour enclosing all the poles $z_1, z_2, \ldots, z_n$ in the finite $z-$plane; since C can be regarded as bounding the outer region in the clockwise direction.

Thus, the residue at the pole $z=\infty$ is $-\frac{1}{2\pi i}\int_C f(z)dz$, *i.e.*, $-\sum_{k=1}^{n} R_{z_k}$. It follows that, *if* $f(z)$ *be analytic, except at a finite number of poles (including the pole at infinity), the sum of the residues at these poles is zero.*

Another theorem for calculating the residue at $z=\infty$ is given in the following article.

5.22. Theorem. *If a single-value function* $f(z)$ *has a pole at* $z=\infty$, *then the residence of* $f(z)$ *at infinity is the negative of the coefficient of* $\frac{1}{z}$ *in the expansion of* $f(z)$ *for values of* z *in the neighbourhood of* $z=\infty$,

If $f(z)$ has a pole of order m at $z=\infty$, then [see § 4.8 (2)]

$$f(z)=\sum_{n=1}^{m} b_n z^n+\sum_{n=0}^{\infty} a_n z^{-n}; \qquad \ldots\ldots(1)$$

$\therefore$ the required residue $=-\frac{1}{2\pi i}\int_C f(z)dz$, where C is a closed contour enclosing all other poles (see § 5.21). Now, the only term in the integrand whose integral around C is not zero is the term in $\frac{1}{z}$.

Hence, the required residue $=-\frac{1}{2\pi i}\int_C a_1 z^{-1}dz=-a_1$. $\ldots\ldots(2)$

If $f(z)$ is analytic at $z=\infty$, *i. e.*, if $b_1=b_2=\ldots=b_m=0$, $f(z)$ still has a residue at infinity, which is zero only if $a_1=0$.

If $z=\infty$ is a regular point of $f(z)$, it is obvious from (1), that

$$-a_1 = -\operatorname*{Lt}_{z\to\infty}[zf(z)], \qquad \ldots\ldots(3)$$

provided $a_0=0$.

Hence, if $\operatorname*{Lt}_{z\to\infty}[-z\,f(z)]$ has a definite finite value, that value is the residue of $f(z)$ at $z=\infty$.

Note. The fact that a function $f(z)$ has a residue at ∞ has nothing to do with the presence of a pole at ∞ ; *i.e.*, $f(z)$ can have a non-zero residue at ∞ irrespective of whether $f(z)$ has a pole at ∞ or not.

Ex. 1. *Show that the residues of* $\dfrac{z}{(z-a)(z-b)}$ *and* $\dfrac{z^3-z^2+1}{z^3}$ *at infinity are —1 and 1 respectively.*

Both these functions are analytic at infinity. For the function $f(z)=\dfrac{z}{(z-a)(z-b)}$, $\operatorname*{Lt}_{z\to\infty}[-z\,f(z)]$ exists and is equal to -1 ; hence, the residue of $f(z)$ at infinity is -1.

For the other function $\operatorname*{Lt}_{z\to\infty}[-z\,f(z)]$ does not exist. Writing it as $1-\dfrac{1}{z}+\dfrac{1}{z^3}$, we have the residue at infinity $=1$, by § 5.22.

Ex. 2. *Evaluate the residues of* $\dfrac{z^3}{(z-1)(z-2)(z-3)}$ *at 1, 2, 3, and infinity, and show that their sum is zero.*

Let $f(z)=\dfrac{z^3}{(z-1)(z-2)(z-3)}$, then $f(z)$ has simple poles at 1, 2, and 3. Hence, by § 5.1 (4)

$$\text{residue at } 1 = \operatorname*{Lt}_{z\to1}[(z-1)f(z)] = \operatorname*{Lt}_{z\to1}\frac{z^3}{(z-2)(z-3)} = \tfrac{1}{2},$$

$$\text{residue at } 2 = \operatorname*{Lt}_{z\to2}[(z-3)f(z)] = \operatorname*{Lt}_{z\to2}\frac{z^3}{(z-1)(z-3)} = -8,$$

$$\text{and residue at } 3 = \operatorname*{Lt}_{z\to3}[(z-3)f(z)] = \operatorname*{Lt}_{z\to3}\frac{z^3}{(z-1)(z-2)} = \frac{27}{2}.$$

Now $\underset{z\to\infty}{\text{Lt}}\ [-zf(z)]$ does not exist. The expansion of $f(z)$ in the neighbourhood of $z=\infty$ is obtained as follows:

$$f(z)=\left(1-\frac{1}{z}\right)^{-1}\left(1-\frac{2}{z}\right)^{-1}\left(1-\frac{3}{z}\right)^{-1}=1+\frac{6}{z}+\text{ higher powers}$$

of $\frac{1}{z}$.

Hence, by § 5.22, the residue of $f(z)$ at infinity $=-6$.
The sum of the residue of $f(z)$ at 1, 2, 3, and ∞

$=\frac{1}{2}-8+\frac{27}{2}-6=0$. The result in § 5. 21 is thus verified.

5.3. Evaluation of Real Definite Integrals. Cauchy's residue theorem renders a great service in the evaluation of certain type of real definite integrals which can also be evaluated by other means, though usually not so simply. In some cases, however, Cauchy's method is the more difficult, *e.g.*, $\int_0^\infty e^{-x^2}dx$ cannot be evaluated by Cauchy's method.

The important types of real definite integrals are discussed in the articles that follow. The methods to be employed in any particular case will be illustrated by typical examples.

We give below a theorem which will be found useful in many cases.

Theorem. *If* $\underset{z\to z_0}{\text{Lt}}\ [(z-z_0)f(z)]=k$, *a constant and if* C *is an arc,* $\theta_1 \leqslant \text{amp}\,(z-z_0) \leqslant \theta_2$, *of the circle* $|z-z_0|=r$, *then*

$$\underset{r\to 0}{\text{Lt}} \int_C f(z)dz = i(\theta_2-\theta_1)k.$$

Given ε, we can find $\delta(\varepsilon)$ such that

$$|(z-z_0)f(z)-k| < \varepsilon \text{ for } |z-z_0| < \delta.$$

Thus, taking $r<\delta$, we have $(z-z_0)f(z)=k+\eta_1$ on C,
where $|\eta_1| < \varepsilon$;

$$\therefore \quad \int_C f(z)dz = \int_C \frac{k+\eta_1}{z-z_0}\,dz = i\int_{\theta_1}^{\theta_2}(k+\eta_1)d\theta,$$

where $z-z_0=re^{i0}$,

i.e., the integral $= i(0_2-0_1)k+i\int_{\theta_1}^{0_2}\eta d\theta.$

Hence,

$$\left|\int_C f(z)dz-i(\theta_2-\theta_1)k\right| \leqslant |\eta|\,(0_2-0_1)$$

$$<\varepsilon\,(0_2-0_1),$$

i. e., $\underset{r\to 0}{\text{Lt}}\int_C f(z)dz = i(0_2-0_1)k.$

Note. If $f(z)$ has a simple pole at $z=z_0$, k is the residue of $f(z)$ at z_0.

5.4. Integration Round the Unit Circle.

An integral of the type $\int_0^{2\pi} f(\cos 0, \sin 0)d0$, where the integrand is a rational function of $\sin\theta$ and $\cos 0$, can be evaluated by writing $e^{i0}=z$.

Since $\cos 0=\frac{1}{2}\left(z+\frac{1}{z}\right)$, $\sin 0=\frac{1}{2i}\left(z-\frac{1}{z}\right)$, $\frac{dz}{iz}=d0$; the integral takes the form $\int_C F(z)dz$, where $F(z)$ is a rational function of z and C is the unit circle $|z|=1$. Hence, by § 5.2, *the integral is equal to $2\pi i$ times the sum of the residues of $F(z)$ at those of its poles which are inside C.*

The work of evaluation of the residues of $F(z)$ may often be considerably simplified by preliminary manipulation of the integral, before introducing the complex variable.

Ex. 1. *Evaluate.* $\int_0^{\pi}\frac{ad\phi}{a^2+\cos^2\phi}$, *where a is positive.*

$$\text{Let } I=\int_0^{\pi}\frac{ad\phi}{a^2+\cos^2\phi}=\int_0^{\pi}\frac{2ad\phi}{1+2a^2-\cos^2 2\phi}$$

$$= \int_0^{2\pi} \frac{a d\theta}{1+2a^2+\cos\theta}, \quad \text{where } \theta = 2\phi.$$

Putting $e^{\theta i} = z$, $I = \int_C \frac{-2aidz}{z^2+2z(1+2a^2)+1}$, where C denotes the circle $|z| = 1$. The integral has simple poles at the points

$$-(1+2a^2)+2a\sqrt{1+a^2} \text{ and } -(1+2a^2)-2a\sqrt{1+a^2}.$$

If we take the positive square root, the latter point lies outside C, the former within C; since

$$-(1+2a^2) - 2a\sqrt{1+a^2} = -(a+\sqrt{1+a^2})^2 < -1, \ (\because a > 0).$$

The residue at the pole within C, by §5.1(3)

$$= \left[\frac{-2ai}{\frac{d}{dz}\{z^2+2z(1+2a^2)+1\}} \right]_{z=-(1+2a^2)+2a\sqrt{1+a^2}}$$

$$= \frac{-i}{2\sqrt{1+a^2}}; \quad \left[\because z+(1+2a^2) = 2a\sqrt{1+a^2} \right].$$

Hence, $I = \frac{2\pi i \times -i}{2\sqrt{1+a^2}} = \frac{\pi}{\sqrt{1+a^2}}$.

Note. The integral $I_1 = \int_0^{\pi} \frac{a d\phi}{a^2+\sin^2\phi}$ can be similarly evaluated. *(Agra 1945)*

Since $\int_0^{\pi/2} \frac{a d\phi}{a^2+\sin^2\phi} = \int_0^{\pi/2} \frac{a d\phi}{a^2+\sin^2(\pi/2-\phi)}$

$$= \int_0^{\pi/2} \frac{a d\phi}{a^3+\cos^2\phi},$$

I_1 shall have the same value as I.

Ex. 2. *Evaluate* $\int_0^{2\pi} e^{-\cos\theta}\cos(n\theta+\sin\theta)d\theta$, *where* n *is a positive integer.*

Consider $I = \int_0^{2\pi} e^{-\cos\theta} \{\cos(n\theta+\sin\theta) - i\sin\theta(n\theta+\sin\theta)\} d\theta$

$$= \int_0^{2\pi} e^{-\cos\theta - i(n\theta+\sin\theta)} . d\theta = \int_0^{2\pi} e^{-(\cos\theta+i\sin\theta)} e^{n\theta i} d\theta.$$

Putting $e^{\theta i}=z$, we have

$I=\int e^{-z}\cdot\frac{1}{z^n}\cdot\frac{dz}{iz}=\int_C \frac{e^{-z}}{iz^{n+1}}dz$, where C is the circle $|z|=1$.

The origin is a pole of order $n+1$ of the integrand, and the residue there is $\frac{1}{\lfloor n}\left[\frac{d^n}{dz^n}\left(\frac{e^{-z}}{i}\right)\right]_{z=0}=\frac{(-1)^n}{i\lfloor n}$.

Since the integrand has no other singularity within C.

$$I=2\pi i\times\frac{(-1)^n}{i\lfloor n}=\frac{2\pi(-1)^n}{\lfloor n}.$$

Hence, equating real parts, we have

$$\int_0^{2\pi} e^{-\cos\theta}\cos(n\theta+\sin\theta)d\theta=\frac{2\pi(-1)^n}{\lfloor n}.$$

5.5. The evaluation of $\int_{-\infty}^{\infty} f(x)dx$.

When $f(x)=\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x, the integral $\int_{-\infty}^{\infty} f(x)dx$ exists only if $Q(x)$ does not vanish for any real value of x and the degree of $P(x)$ is less than the degree of $Q(x)$ by at least two. The integral is taken along a closed contour C consisting of the real axis from $-R$ to R and C_R, the upper-half of the circle $|z|=R$ (see fig. 28), where R is so chosen that $f(z)$ has no poles on the semi-circle. By Cauchy's residue theorem, we have

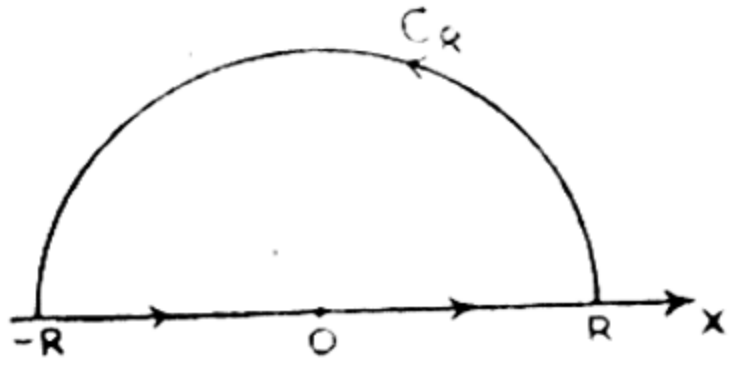


Fig. 28

$$2\pi i\sum R_{z_k}=\int_C f(z)dz=\int_{-R}^{R} f(x)dx+\int_{C_R} f(z)dz,$$

where z_k are the poles of $f(z)$ within C.

Now, if $P(z)$ and $Q(z)$ are polynomials in z of degrees n and m respectively such that $m-n\geqslant 2$, $|z\,f(z)|<\varepsilon$ for sufficiently large values of R.

Hence, $\left| \int_{C_R} f(z)\mathrm{d}z \right| \leqslant \pi\epsilon$, and

$$\therefore \quad \underset{R\to\infty}{\mathrm{Lt}} \int_{C_R} f(z)dz = 0.$$

It follows that $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum R_{z_k}$, where the summation now extends to residues of $f(z)$ at all its poles in the upper-half plane.

We calculate here $\underset{R\to\infty}{\mathrm{Lt}} \int_{-R}^{R} f(x)dx$, which is equal to $\int_{-\infty}^{\infty} f(x)dx$ only when the latter integral exists in the ordinary sense, *i.e.*, when each of the integrals $\int_{-\infty}^{0} f(x)dx, \int_{0}^{\infty} f(x)dx$ converges separately. However, $\int_{-\infty}^{\infty} f(x)dx$ may not exist in the ordinary sense while $\underset{R\to\infty}{\mathrm{Lt}} \int_{-R}^{R} f(x)dx$ exists. In this case the latter integral is called the *Cauchy principal value* of the former, and we write.

$$\underset{R\to\infty}{\mathrm{Lt}} \int_{-R}^{R} f(x)dx = \mathrm{P.V.} \int_{-\infty}^{\infty} f(x)dx.$$

Ex. 1. *Evaluate, by contour integration,* $\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)}$.

Let $I = \int_C \frac{z^2dz}{(z^2+1)(z^2+4)}$, where C is the contour consisting of the real axis from $-R$ to R and C_R, the upper-half of the circle $|z| = R$.

The singularities within the contour C, when R is large, consist of simple poles at the points $z=i$ and $z=2i$.

The residue of integrand at $i = \underset{z\to i}{\text{Lt}} \dfrac{(z-i)z^2}{(z^2+1)(z^2+4)} = -\dfrac{1}{6i}$,

and the residue at $2i = \underset{z\to 2i}{\text{Lt}} \dfrac{(z-2i)\,z^2}{(z^2+1)(z^2+4)} = \dfrac{1}{3i}$.

Hence, by Cauchy's residue theorem

$$I = 2\pi i\left(-\frac{1}{6i}+\frac{1}{3i}\right) = \frac{\pi}{3},$$

where $I = \int_{-R}^{R} \dfrac{x^2dx}{(x^2+1)(x^2+4)} + \int_{C_R} \dfrac{z^2dz}{(z^2+1)(z^2+4)}$.

Now, $\left| \int_{C_R} \dfrac{z^2dz}{(z^2+1)(z^2+4)} \right| \leqslant \int_{C_R} \dfrac{|z|^2\,|dz|}{|z^2+1|\,|z^2+4|}$

$$\leqslant \frac{R^2.\pi R}{(R^2-1)(R^2-4)},$$

$[\because |z^2+1| \geqslant |z|^2-1$, and $|z^2+4| \geqslant |z|^2-4.]$

which $\to 0$ as $R\to\infty$.

$$\therefore \underset{R\to\infty}{\text{Lt}} \int_{-R}^{R} \frac{x^2dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}.$$

Since the integrand behaves like $\dfrac{1}{x^2}$ for large values of x, the integral $\int_{-\infty}^{\infty} \dfrac{x^2dx}{(x^2+1)(x^2+4)}$ exists in the ordinary sense and has, therefore, the value $\dfrac{\pi}{3}$.

Ex. 2. *Use the residue theory to show that*

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right),$$

$(a>b>0)$.

Let us conside $I = \int_C f(z)\,dz$, where $f(z) = \dfrac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$, taken round the contour of §5.5. When $R>a$, $f(z)$ has two simple poles within C, namely $z=ia$ and $z=ib$.

The residue of $f(z)$ at $ia = \underset{z \to ia}{\text{Lt}} \dfrac{(z-ia)e^{iz}}{(z^2+a^2)(z^2+b^2)} = \dfrac{e^{-a}}{-2ai(a^2-b^2)}$;

and the residue at $ib = \dfrac{e^{-b}}{2bi(a^2-b^2)}$ [on interchanging a and b].

Hence, by Cauchy's residue theorem

$$I = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

$$= 2\pi i \left[\frac{e^{-a}}{-2ai(a^2-b^2)} + \frac{e^{-b}}{2bi\,(a^2-b^2)}\right]$$

$$= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{b}\right). \qquad \text{.....(1)}$$

Now $\left| \int_{C_R} f(z)\,dz \right| \leqslant \int_{C_R} |f(z)|\,|dz|$

$$\leqslant \frac{\pi R}{(R^2-a^2)(R^2-b^2)}; \text{ since } |a^{iz}| = |e^{-v}|$$

$\leqslant 1$, when $y > 0$, $|z^2+a^2| \geqslant |z|^2 - a^2$, $|z^2+b^2| \geqslant |z|^2 - b^2$.

$$\therefore \quad \int_{C_R} f(z)\,dz \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

Hence, $\int_{-\infty}^{\infty} \dfrac{e^{ix}\,dx}{(x^2+a^2)(x^2+b^2)}$, which obviously exists in the ordinary sense, is equal to the vaule in (1). Now, equating real parts, we find that

$$\int_{-\infty}^{\infty} \frac{\cos x\,dz}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

Note. If $a=b$, then $f(z)$ will have a double pole at $z=ia$.

5.51. Indented Semi-circluar Contour.

If $Q(x)$ in § 5.5 has non-repeated real linear factors, *i.e.*, $f(z)$ has simple poles on real axis, the principal value of $\int_{-\infty}^{\infty} f(x)dx$ exists.* To evaluate

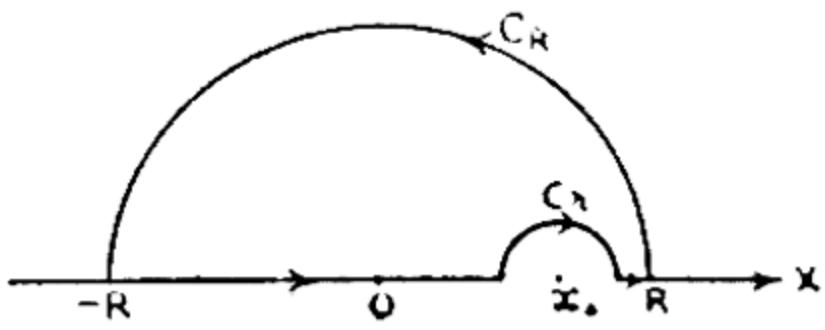


Fig. 29

*See Bromwich, *Theory of Infinite Series*, p. 415.

the integral in this case, we modify the contour of § 5.5 in the following way. We delete the neighbourhood of a pole x_0 on the real axis by iuserting a small circle C_r, $|z-x_0|=r$, (see fig. 29). A contour modified in this way is said to be **indented**. The integral round the indented contour is calculated by Cauchy's residue theorem and then the radius of each indentation is made to tend to zero. Since $\int f(z)dz$ taken along the semi-circle C_r, $|z-z_0|=r$, is equal to πi times the residue at x_0 [§ 5.3, Th.], it follows that the value of $\int_{-\infty}^{\infty} f(x)dx$ is $2\pi i$ times the sum of the residues of $f(z)$ at its poles within the indented contour *plus* πi times the sum of the residues of $f(z)$ at its poles on the real axis.

Ex. 1. *Prove that, if* $a > 0$,

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{\cos x\, dx}{a^2-x^2} = \frac{\pi \sin a}{a}.$$

Let $f(z)=\dfrac{e^{iz}}{a^2-z^2}$, than $f(z)$ has two simple poles on the real axis, namely $z=\pm a$.

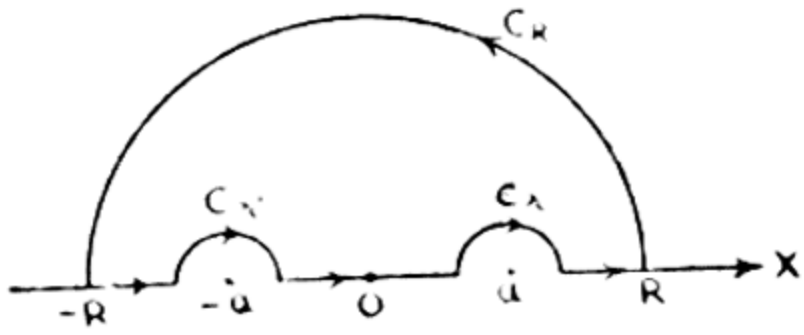


Fig. 30

We consider $I=\int_C f(z)dz$,

where C is the contour of § 5.5 indented at $z=a$ and $z=-a$. Let the radii of indentation at a and $-a$ be respectively r and r', then within the contour thus modified (see fig 30) $f(z)$ has no singularity.

$$\therefore I=\int_{-R}^{-a-r'} f(x)dx-\int_{C_r'} f(z)dz+\int_{-a+r'}^{a-r} f(x)dx$$

$$-\int_{C_r} f(z)dz+\int_{a+r}^{R} f(x)dx+\int_{C_R} f(z)dz=0, \qquad \ldots\ldots(1)$$

Now, $z=-a$ and $z=a$ are simple poles of $f(z)$;

$\therefore$ the residue of $f(z)$ of $(z=-a)=\dfrac{e^{-ia}}{2a}$,

and the residue at $(z=a)=\dfrac{e^{ia}}{2a}$;

Hence, $\underset{r'\to 0}{\text{Lt}} \int_{C_{r'}} f(z)dz = \pi i \times \frac{e^{-ia}}{2a}$,

and $\underset{r\to 0}{\text{Lt}} \int_{C_r} f(z)dz = \pi i \times \frac{e^{ia}}{2a}$. [§5.3, Th.]

Also, $\left| \int_{C_R} f(z)dz \right| \leqslant \int_{C_R} \frac{|e^{iz}|}{|a^2-z^2|} |dz| \leqslant \frac{\pi R}{R^2-a^2}$,

since $|e^{iz}| e^{-y} \leqslant 1$ for $y \leqslant 0$, and $|a^2-z^2| \geqslant |z|^2 - a^2$ ($\because |z| > a$);

$\therefore \int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

Making $R \to \infty$, $r \to o$, $r' \to 0$, we have

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{e^{ix}dx}{a^2-x^2} = \frac{-\pi i}{2a}(e^{ia} - e^{-ia}) = \frac{\pi \sin a}{a}. \qquad \ldots\ldots(2)$$

Equating real parts, we obtain the required result

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{\cos x\, dx}{a^2-x^2} = \frac{\pi \sin a}{a}.$$

Note 1. If we equate the inaginary parts in (2), we get

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{\sin x\, dx}{a^2-x^2} = 0.$$

Note 2. In (i), $\int_{C_{r'}}$ and $\int_{C_r}$ are taken with negative sign, since we have defined contour integration in the counterclockwise direction to be positive.

Ex. 2. *Show that, if* $a \geqslant 0$, $b \geqslant 0$, *then*

$$\int_0^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2} dx = \pi(b-a). \qquad \textit{(Agra 1948)}$$

Let $f(z) = \frac{e^{2iaz} - e^{2ibz}}{z^2}$, and consider $I = \int_C f(z)dz$,

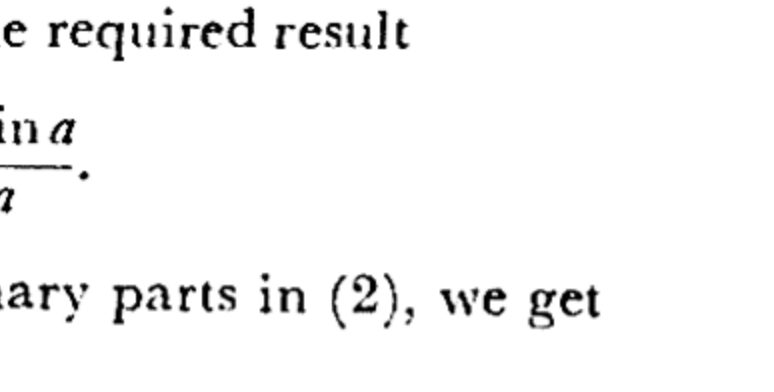


Fig 31

taken round a contour consisting of the upper-half of a large circle

$|z|=R$, the upper-half of a small circle $|z|=r$ and the parts of the real axis joining their ends (see Fig. 31). There is no singularity within the contour; hence, by Cauchy's residue theorem

$$I\equiv \int_{-R}^{-r} f(x)dx - \int_{C_r} f(z)dz + \int_{r}^{R} f(x)dx + \int_{C_R} f(x)dz = 0.$$

Now, $z=0$ is a simple pole of $f(z)$;

$$\left[\because f(z)=\frac{2i(a-b)z-2(a^2-b^2)z^2-\cdots}{z^2}\right]$$

$\therefore$ residue at $0=2i(a-b)$.

Hence, $\underset{r\to 0}{\text{Lt}} \int_{C_r} f(z)dz=\pi i\times 2i(a-b)=2\pi(b-a)$.

Also, $$\left|\int_{C_R} f(z)dz\right| \leqslant \int_{C_R} \frac{|e^{2iaz}-e^{2ibz}|}{|z|^2}\,|dz|$$

$$\leqslant \frac{2\pi R}{R^2},$$

since $|e^{2iaz}-e^{2ibz}| \leqslant |e^{2iaz}|+|e^{2ibz}| = e^{-2ay}+e^{-2by}$

$\leqslant 2$, when $y\geqslant 0$ ($\because a\geqslant 0,\ b\geqslant 0$) ;

$$\therefore \int_{C_R} f(z)dz\to 0 \text{ as } R\to\infty.$$

Hence, making $R\to\infty$, $r\to 0$, we have

$$\text{P. V.}\int_{-\infty}^{\infty} \frac{e^{2iax}-e^{2ibx}}{z^2}dx=2\pi(b-a).$$

Equating real parts, we have $\int_{-\infty}^{\infty} \frac{\cos 2ax-\cos 2bx}{x^2}dx=2\pi(b-a)$,

or $$\int_{0}^{\infty} \frac{\cos 2ax-\cos 2bx}{x^2}dx=\pi(b-a).$$

5.52. Jordan's Inequality. As θ increases from 0 to $\frac{\pi}{2}$, $\cos\theta$ decreases steadily. Hence, the mean ordinate of the graph of $y=\cos x$ over the range $0\leqslant x\leqslant 0$, *i.e.*, $\frac{1}{0}\int_0^0 \cos x\,dx=\frac{\sin 0}{0}$ also decreases steadily ;

$$\therefore \quad \frac{2}{\pi} \leqslant \frac{\sin\theta}{\theta} \leqslant 1, \text{ when } 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

This is known as **Jordan's inequality** and is very useful in the evaluaton of definite integrals of the type $\int_{-\infty}^{\infty} e^{miz} f(x)dx$, where $f(x)$ is a rational function $\frac{P(x)}{Q(x)}$ such that the degree of $P(x)$ is less than the degree of $Q(x)$ by at least one.

Ex. 1. *Show that, if $k>0$, $a>0$, then*

$$\int_0^{\infty} \frac{x \sin ax}{x^2+k^2} dx = \frac{\pi e^{-ka}}{2}. \qquad \textit{(Agra, 1940,'49)}$$

Let $f(z) = \frac{ze^{iaz}}{z^2+k^2}$ and consider $I = \int_C f(z)\, dz$ taken round the contour of §5.5. The only pole within the contour is ik ($\because k>0$), the residue at which is $\underset{z \to ik}{\text{Lt}} \frac{(z-ik)ze^{iaz}}{z^2+k^2} = \frac{e^{-ka}}{2}$.

Hence, by Cauchy's residue theorem

$$I \equiv \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i \times \frac{e^{-ka}}{2} = \pi i e^{-ka}.$$

Now, $\left| \int_{C_R} f(z)\, dz \right| \leqslant \int_{C_R} \frac{|z|\,|e^{iaz}|}{|z^2+k^2|} |dz|$

$$\leqslant \frac{R^2}{R^2-k^2} \int_0^{\pi} e^{-aR\sin\theta}\, d\theta, \text{ where } z = Re^{i\theta}$$

$$\leqslant \frac{2R^2}{R^2-k^2} \int_0^{\pi/2} e^{-2aR\theta/\pi}\, d\theta, \text{ by Jordan's inequality}$$

$$i.e., \quad \leqslant \frac{\pi R}{a(R^2-k^2)} \left| -e^{-2aR\theta/\pi} \right|_0^{\pi/2}$$

$$= \frac{\pi R(1-e^{-aR})}{a(R^2-k^2)},$$

which $\longrightarrow 0$ as $R \longrightarrow \infty$, if $a>0$.

Thus, $\underset{R\to\infty}{\text{Lt}} \int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+k^2}\, dz = \pi i e^{-ka}$.

Equating the imaginary parts, we have

$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2+k^2} = \pi e^{-ka},$$

$$\text{or} \quad \int_{0}^{\infty} \frac{x \sin ax}{x^2+k^2}\, dx = \frac{\pi}{2} e^{-ka}.$$

Ex. 2. *Apply the calculus of residues to evalute*

$$\int_{0}^{\infty} \frac{\sin mx}{x}\, dx, \text{ where } m>0. \qquad (Agra\ 1940, '55)$$

Let $f(z)=\frac{e^{miz}}{z}$ and consider $I=\int_C f(z)\, dz$, taken round the contour of fig. 31. There is no singularity within the contour ; hence by Cauchy's residue theorem

$$I \equiv \int_{-R}^{-r} f(x)\, dx - \int_{C_r} f(z)\, dz + \int_{r}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 0.$$

Now, $z=0$ is a simple pole of $f(z)$;

$$\therefore \quad \text{residue at } 0 = \underset{z\to 0}{\text{Lt}} \frac{ze^{miz}}{z} = 1.$$

Hence, $\underset{r\to 0}{\text{Lt}} \int_{C_r} f(z)\, dz = \pi i$.

Also, $$\left| \int_{C_R} f(z)\, dz \right| \leqslant \int_{C_R} \frac{|e^{miz}|}{|z|} |dz|$$

$$= \int_{0}^{\pi} e^{-mR \sin\theta} d\theta, \text{ where } z=Re^{i\theta}$$

$$\leqslant 2\int_{0}^{\pi/2} e^{-2mR\theta/\pi} d\theta, \text{ by Jordan's inequality}$$

$$= \frac{\pi}{mR}(1-e^{-mR}),$$

which $\to 0$ as $R\to\infty$, if $m>0$.

Hence, making $R\to\infty$, $r\to 0$, we have

$$\text{P. V.}\int_{-\infty}^{\infty}\frac{e^{mix}}{x}\,dx=\pi i. \qquad \text{.....(1)}$$

Equating imaginary parts, we get $\int_{-\infty}^{\infty}\frac{\sin mx}{x}\,dx=\pi$,

or $\int_{0}^{\infty}\frac{\sin mx}{x}\,dx=\frac{\pi}{2}$.

Note. If we equate real parts in (1), we get

$\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos mx}{x}\,dx=0$; the P.V. is necessary, since the integrand becomes infinite at origin, whereas in the integral $\int_{-\infty}^{\infty}\frac{\sin mx}{x}\,dx$, the integrand $\to m$ as $x\to 0$.

5.6. Evaluation of $\int_{0}^{\infty}x^{a-1}\,f(x)\,dx$ $(0<a<1)$. An integral of this type can be evaluated by contour integration if $f(z)$ is a rational function either having no poles or else only simple poles on the real axis and if $x^a f(x)\to 0$ as $x\to 0$ and also as $x\to\infty$; for, then $\int_{0}^{\infty}x^{a-1}f(x)dx$ converges at the upper and lower limits of integration and possesses a Cauchy principal value. The different methods of evaluating such integrals are illustrated in the following example.

Ex. 1. *Apply the calculus of residnes to evaluate*

$$\int_{0}^{\infty}\frac{t^{a-1}}{1+t}dt,\ (0<a<1).$$

Method I. Putting $t=e^x$, we have

$$\int_{0}^{\infty}\frac{t^{a-1}}{1+t}\,dt$$

$$=\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx.$$

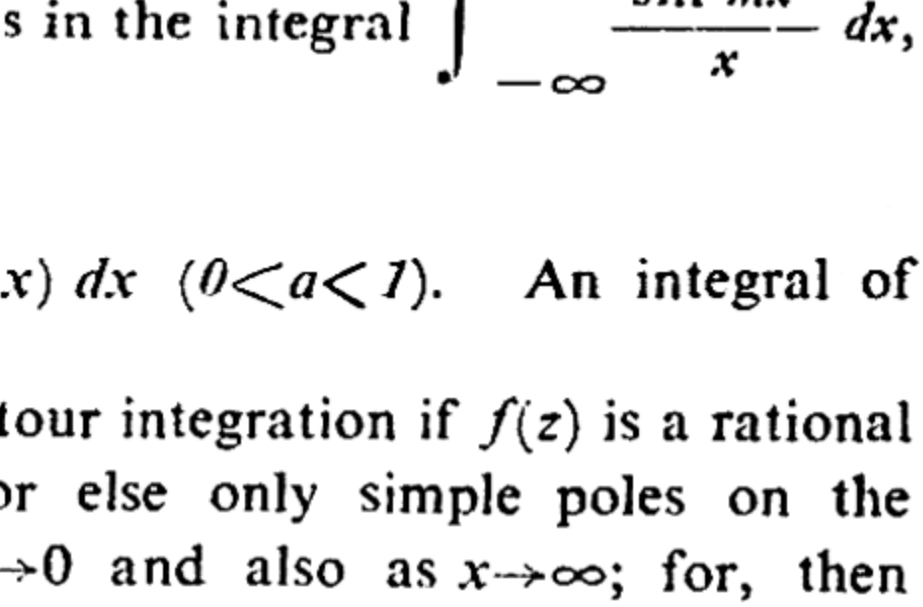


Fig. 32

Let $f(z)=\dfrac{e^{az}}{1+e^{z}}$ and consider $I=\int_C f(z)\,dz$, taken round the contour consisting of the $x-$axis and the lines $x=\pm R, y=2\pi$. The only singularity within the contour is a simple pole at $z=\pi i$ and the residue there is $\left[\dfrac{e^{az}}{\dfrac{d}{dz}(1+e^{z})}\right]_{z=\pi i}=-e^{a\pi i}$.

Hence, by Cauchy's residue theorem

$$I\equiv\int_{-R}^{R} f(x)\,dx+\int_{0}^{2\pi} f(R+iy).\,i dy$$
$$+\int_{R}^{-R} f(x+2\pi i)\,dx+\int_{2\pi}^{0} f(-R+iy).\,i dy$$

$$=-e^{a\pi i}\times 2\pi i.$$

Now, $\left|\int_0^{2\pi} f(R+iy).\,i\,dy\right|\leqslant\int_0^{2\pi}\dfrac{|e^{aR+iay}|}{|1+e^{R+iy}|}\,dy$

$$\leqslant\frac{2\pi\,e^{aR}}{e^{R}-1},$$

$$(\therefore\ |e^{R+iy}+1|\geqslant e^{R}-1)$$

which $\longrightarrow 0$ as $R\longrightarrow\infty$, if $a<1$.

Similarly, $\left|\int_0^{2\pi} f(-R+iy).\,idy\right|\leqslant\dfrac{2\pi\,e^{-aR}}{1-e^{-R}}$,

which $\longrightarrow 0$ as $R\longrightarrow\infty$, if $a>0$.

Hence, $\underset{R\to\infty}{\text{Lt}}\left[\int_{-R}^{R} f(x)\,dx-\int_{-R}^{R} e^{2a\pi i}f(x)\,dx\right]$

$$=-2\pi i e^{a\pi i},$$

or $\displaystyle\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}\,dx=\frac{-2\pi i e^{a\pi i}}{1-e^{2a\pi i}}=\pi.\ \frac{2i}{e^{a\pi i}-e^{-a\pi i}}$

$$=\frac{\pi}{\sin a\pi};\qquad (Agra,\ 41,\ '51)$$

so that $\int_0^\infty \frac{t^{a-1}}{1+t}\,dt=\frac{\pi}{\sin a\pi}$.

Method II. Consider

$$I_1=\int_{C_1}\phi(z\}dz,$$

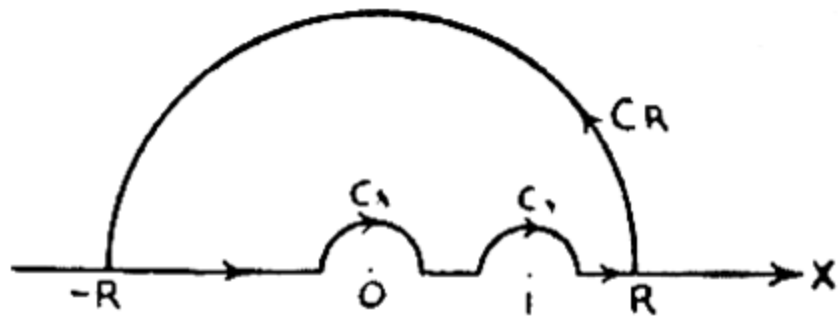


Fig. 33

where $\phi(z)=\frac{z^{a-1}}{1-z}$, z^{a-1} having its principal value*, and C_1 is a large semi-circular contour indented at $z=0$ and $z=1$, the radii of indentation being r and r' respectively (see fig. 33).

There is no singularity within the contour ; hence, by Cauchy's residue theorem

$$I_1\equiv\int_R^r e^{\pi i}\,\phi\,(xe^{\pi i})\,dx-\int_{C_r}\phi\,(z)\,dz$$
$$+\int_r^{1-r'}\phi\,(x)\,dx-\int_{C_{r'}}\phi\,(z)\,dz+\int_{1-r'}^{R}\phi(x)\,dx$$
$$+\int_{C_R}\phi(z)\,dz=0. \qquad \ldots\ldots(1)$$

Now, $\left|\int_{C_R}\phi(z)\,dz\right|\leqslant\int_{C_R}\frac{|z|^{a-1}}{|z-1|}\,|dz|$

$$\leqslant\frac{\pi R^a}{R-1},\ \text{which}\longrightarrow 0\ \text{as}\ R\longrightarrow\infty,$$

since $a<1$.

Again, $\underset{z\to 0}{\text{Lt}}\ [z\phi\,(z)]=\underset{z\to 0}{\text{Lt}}\ \frac{z^a}{1-z}=0$, since $a>0$;

* a being fractional, z^{a-1} is a many-valued function. z^{a-1} $e^{(a-1)\,\text{Log}\,z}$ has its principal value when Log z has its principal value. On the negative real axis, $z^{a-1}=(xe^{\pi i})^{a-1}$ and on the positive real axis $z^{a-1}=x^{a-1}$. $z=0$ is a branch point of $f(z)$.

$$\therefore \quad \underset{r\to 0}{\text{Lt}} \int_{C_r} \phi(z)\, dz = 0.$$

Also, $\underset{z\to 1}{\text{Lt}}\ [(z-1)\,\phi(z)] = -1$; $\therefore \underset{r'\to 0}{\text{Lt}} \int_{C_{r'}} \phi(z)\, dz = -\pi i.$

Hence, making $R \longrightarrow \infty$, $r \longrightarrow 0$, $r' \longrightarrow 0$, we have

$$\int_{\infty}^{0} \frac{e^{a\pi i}\ x^{a-1}}{1+x}\, dx + \text{P.V.} \int_{0}^{\infty} \frac{x^{a-1}}{1-x} dx + \pi i = 0. \qquad \ldots\ldots(2)$$

Equating imaginary parts, we have

$$\sin a\pi \int_{0}^{\infty} \frac{x^{a-1}}{1+x}\, dx = \pi,$$

$$\text{or} \quad \int_{0}^{\infty} \frac{x^{a-1}}{1+x}\, dx = \frac{\pi}{\sin a\pi}. \qquad \ldots\ldots(3)$$

(Agra 1935, '47, '48 ; Vikram 1960)

If we equate real parts in (2) we get

$$\text{P.V.} \int_{0} \frac{x^{a-1}}{1-x} dx = \cos a\pi \int_{0}^{\infty} \frac{x^{a-1}}{1+x}\, dx$$

$$= \pi \cot a\pi. \ [\because \quad \text{of (3)}] \qquad \ldots\ldots(4)$$

(Vikram 1959)

Method III. Let $\psi(z) = \dfrac{z^{a-1}}{1+z}$, and consider $I_2 = \int_{C_2} \psi(z) dz$, where z^{a-1} has its principal value and C_2 is a contour consisting of the arcs of circles of radii R and r, and the straight lines joining their end points along the real axis (see fig. 34). The only singularity within the contour is a simple pole at $z = -1 = e^{i\pi}$, the residue at which is

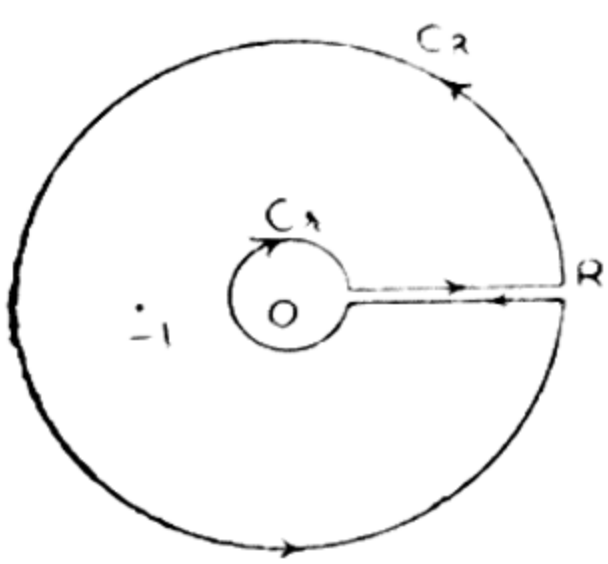


Fig. 34

$$\underset{z\to e^{\pi i}}{\text{Lt}} \frac{(z+1)z^{a-1}}{1+z} = e^{(a-1)\pi i}$$; hence, by Cauchy's residue theorem

$$I_2 \equiv \int_r^R \psi(x)dx + \int_{C_R} \psi(z)dz - \int_r^R \psi(xe^{2\pi i}).e^{2\pi i}dx - \int_{C_r} \psi(z)dz$$

$$= 2\pi i.\, e^{(a-1)\pi i} = -2\pi i e^{a\pi i}.$$

Now $\left| \int_{C_R} \psi(z)dz \right| \leqslant \int_{C_R} \frac{|z|^{a-1}}{|z+1|} |dz|$

$$\leqslant \frac{2\pi R^a}{1-r}, \text{ which} \to 0 \text{ as } R \to \infty, \text{ since } a < 1.$$

So also $\left| \int_{C_R} \psi(z)dz \right| \leqslant \frac{2\pi r^a}{1-r}$, which $\to 0$ as $r \to 0$, since $a > 0$.

Hence, $\lim_{\substack{R\to\infty \\ r\to 0}} \left[\int_r^R \psi(x)dx - \int_r^R \psi(xe^{2\pi i})e^{2\pi i}dx \right] = 2\pi i e^{a\pi i}$,

or $\left(1 - e^{2a\pi i}\right) \int_0^\infty \frac{x^{a-1}}{1+x} dx = -2\pi i e^{a\pi i}$,

or $\int_0^\infty \frac{x^{a-1}}{1+x} dx = \frac{2\pi i}{e^{a\pi i} - e^{-a\pi i}} = \frac{\pi}{\sin a\pi}$.

Ex. 2. *If* $-1 < p < 1$ *and* $-\pi < \lambda < \pi$, *show by contour integration that*

$$\int_0^\infty \frac{x^{-p}dz}{1+2x\cos\lambda+z^2} = \frac{\pi}{\sin x\pi} \cdot \frac{\sin p\lambda}{\sin \lambda}$$

explaining clearly the necessity of conditions on p *and* λ. *(Agra 1943,'44)*

Let $f(z) = \frac{z^{-p}}{1+2z\cos\lambda+z^2} = \frac{z^{-p}}{(z+e^{i\lambda})(z+e^{-i\lambda}}$, and

consider $I = \int_C f(z)dz$, where C is the contour of fig. 34.

$z=0$ is a branch point of $f(z)$, and if $-\pi < \lambda < \pi$, $f(z)$ has no singularity on the positive real axis (excluding origin).

The singularities within the contour consist of simple poles at

$z = -e^{i\lambda} = e^{i(\pi+\lambda)}$, and $z = -e^{-i\lambda} = e^{i(\pi-\lambda)}$;

the amplitudes taken are $\pi+\lambda$ and $\pi-\lambda$, as these are the positive values that lie between 0 and 2π when $-\pi<\lambda<\pi$.

$$\text{Residue of } f(z) \text{ at} -e^{i\lambda} [\text{or, } e^{i(\pi+\lambda)}]$$

$$= \underset{z\to -e^{i\lambda}}{\text{Lt}} \frac{(z+e^{i\lambda})z^{-p}}{(z+e^{i\lambda})(z+e^{-i\lambda})} = \frac{[e^{i(\pi+\lambda)}]^{-p}}{-(e^{i\lambda}-e^{-i\lambda})}$$

$$= -\frac{e^{-ip(\pi+\lambda)}}{2i\sin\lambda}.$$

Similarly, residue at $-e^{-i\lambda} = \dfrac{e^{-ip(\pi-\lambda)}}{2i\sin\lambda}$ (changing λ to $-\lambda$).

Hence, by Cauchy's residue theorem

$$I \equiv \int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_R^r f(xe^{2\pi i}).\ e^{2\pi i}dx$$

$$- \int_{C_r} f(z)dz$$

$$= 2\pi i\left[-\frac{e^{-ip(\pi+\lambda)}}{2i\sin\lambda} + \frac{e^{-ip(\pi-\lambda)}}{2i\sin\lambda}\right]$$

$$= \pi\ e^{-ip\pi}\ (e^{ip\lambda}-e^{ip\lambda})/\sin\lambda$$

$$= 2\pi i\, e^{-ip\pi}\ .\frac{\sin p\lambda}{\sin\lambda}.$$

Now, $\left|\int_{C_R} f(z)dz\right| \leqslant \int_{C_R} \dfrac{|z|^{-p}\,|dz|}{|z+e^{i\lambda}|\,|z+e^{-i\lambda}|}$

$$\leqslant \frac{2\pi R^{1-p}}{(R-1)^2}, \left[\because |z+e^{\pm i\lambda}| \geqslant |z|-1.\right]$$

which $\to 0$ as $R\to\infty$, if $1-p<2$, *i. e.*, if $-1<p$.

Again, $\left|\int_{C_r} f(z)dz\right| \leqslant \dfrac{2\pi r^{1-p}}{(1-r)^2}$, which $\to 0$ as $r\to 0$ if $1-p>0$, *i. e.*, if $p<1$. Thus, the condition $-1<p<1$ is necessary if the two integrals around C_R and C_r are to approach zero as $R\to\infty$ and $r\to 0$.

Hence, $(1-e^{-2p\pi i})\int_0^\infty \frac{x^{-p}dx}{1+2x\cos+x^2}$

$$=2\pi i\, e^{-ip\pi}\ \frac{\sin p\lambda}{\sin \lambda},$$

or $$\int_0^\infty \frac{x^{-p}dx}{1+2x\cos\lambda+x^2}=\frac{2\pi i}{e^{p\pi i}-e^{-p\pi i}}\ \frac{\sin p\lambda}{\sin\lambda}$$

$$=\frac{\pi}{\sin p\pi}\cdot\frac{\sin p\lambda}{\sin\lambda}.$$

Note 1. The result remains unchanged if p is changed to $-p$,

i.e., $\int_0^\infty \frac{x^{p}dx}{1+2\cos\lambda+z^2}=\frac{\pi\sin p\lambda}{\sin\pi\sin\lambda}$, where $-1<p<1$, $-\pi<\lambda<\pi$.

Note 2. That $\lim_{z\to 0}\int_{C_r} f(z)dz=0$ can be shown also by using Theorem of §5.3 ;

for, $\lim_{z\to 0} f(z)=\lim_{z\to 0}\frac{z^{1-p}}{1+2z\cos\lambda+z^2}=0$, if $1-p>0$,

i. e., if $p<1$.

5.7. Rectangular Contours. So far we have been mostly using circular or semi-circular contour [except for Ex. 1, §5.6]. Generally, a large semi-circle, indented or not, is used in evaluating integrals of the type $\int_{-\infty}^{\infty} f(x)(dx)$. It should be noticed, however, that there is no special merit in a semi-circle. In some cases a rectangular contour is more convenient. Improper integrals involving hyperbolic function in the denominator of the integrand are often evaluated by means of a rectangular contour. The rectangular contour is indented when there is a singlarity on the contour as in the case of a semi-circle contour.

Ex. 1. *By integrating* $\frac{e^{az}}{\cosh\pi z}$ *round the rectangle of sides* $x=\pm R$, $y=1$, *show that*

$$\int_0^\infty \frac{\cosh ax}{\cosh \pi x} dx = \tfrac{1}{2}\sec \frac{a}{2}, \text{ where } -\pi < a < \pi.$$

(Agra 1953 ; Vikram 1960)

Let $f(z) = \dfrac{e^{az}}{\cosh \pi z} = \dfrac{2e^{(a+\pi)z}}{e^{2\pi z}+1}$, then a zero of the denominator is given by $e^{2\pi z}+1=0$, or, $e^{2\pi z} = -1 = e^{(2n+1)\pi i}$ $(n=0, 1, 2,\ldots)$.

Thus, the only singularity of $f(z)$ within the given contour is a simple pole at $z=\dfrac{i}{2}$ at which the residue is

$$\left[\frac{e^{az}}{\frac{d}{dz}(\cosh \pi z)}\right]_{z=\frac{i}{2}} = \frac{e^{ai/2}}{\pi i};$$

hence, by Cauchy's residue theorem.

$$I \equiv \int_{-R}^{R} f(x)dx + \int_0^1 f(R+iy).\, idy + \int_R^{-R} f(x+i)dy + \int_1^0 f(-R+iy).\, idy$$

$$= 2\pi i \times \frac{e^{ai/2}}{\pi i} = 2e^{ai/2}.$$

Now, $\left| \int_0^1 f(R+iy).\, idy \right| \leqslant \int_0^1 \frac{2\,|\,e^{(a+\pi)(R+iy)}\,|}{|\,e^{2\pi(R+iy)}+1\,|}\, dy$

$$\leqslant \frac{2e^{(a+\pi)R}}{e^{2\pi R}-1},$$

which $\to 0$ as $R\to\infty$, if $a+\pi<2\pi$, *i.e.*, if $a<\pi$.

Similarly, $\left| \int_0^1 f(-R+iy)idy \right| \leqslant \dfrac{2e^{-(a+\pi)R}}{1-e^{-2\pi R}}$,

which $\to 0$ as $R\to\infty$, if $a+\pi>0$, *i.e.*, if $-\pi<a$.

Hence, $\int_{-\infty}^{\infty} \dfrac{e^{ax}}{\cosh \pi x}dx + e^{ai}\int_{-\infty}^{\infty} \dfrac{e^{ax}}{\cosh \pi x}dx = 2e^{ia/2}$,

$$\text{or} \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x} dx = \frac{2e^{ai/2}}{1+e^{ai}} = \frac{2}{e^{ai/2}+e^{-ai/2}} = \sec \frac{a}{2}. \quad \ldots\ldots(i)$$

Changing x to $-x$, we have

$$\int_{-\infty}^{\infty} \frac{e^{-ax}}{\cosh \pi x} dx = \sec \frac{a}{2}. \quad \ldots\ldots(2)$$

Adding (1) and (2), we have $\int_{-\infty}^{\infty} \frac{\cosh ax}{\cosh \pi x} dx = \frac{1}{2} \sec \frac{a}{2}$,

$$\text{or} \quad \int_{0}^{\infty} \frac{\cosh ax}{\cosh \pi x} dx = \tfrac{1}{2} \sec \frac{a}{2}.$$

Ex. 2. *Integrate* $e^{iaz}/(e^{2\pi z}-1)$, *where a is real, round the rectangle of sides* $x=0$, $x=R$, $y=0$, $y=1$, *indented at 0 and i, and show that*

$$\int_{0}^{\infty} \frac{\sin ax}{e^{2\pi z}-1} dx = \frac{1}{4} \coth \frac{a}{2} - \frac{1}{2a}. \qquad \textit{(Agra 1947)}$$

Let $f(z) = \dfrac{e^{iaz}}{e^{2\pi z}-1}$, and consider

$$I \equiv \int_{r}^{R} f(x)\, dx + \int_{0}^{1} f(R+iy).\, i dy + \int_{R}^{r'} f(x+i) dx$$

$$= \int_{C_{r'}} f(z)\, dz + \int_{1-r'}^{r} f(iy).\, i dy - \int_{C_r} f(z)\, dz,$$

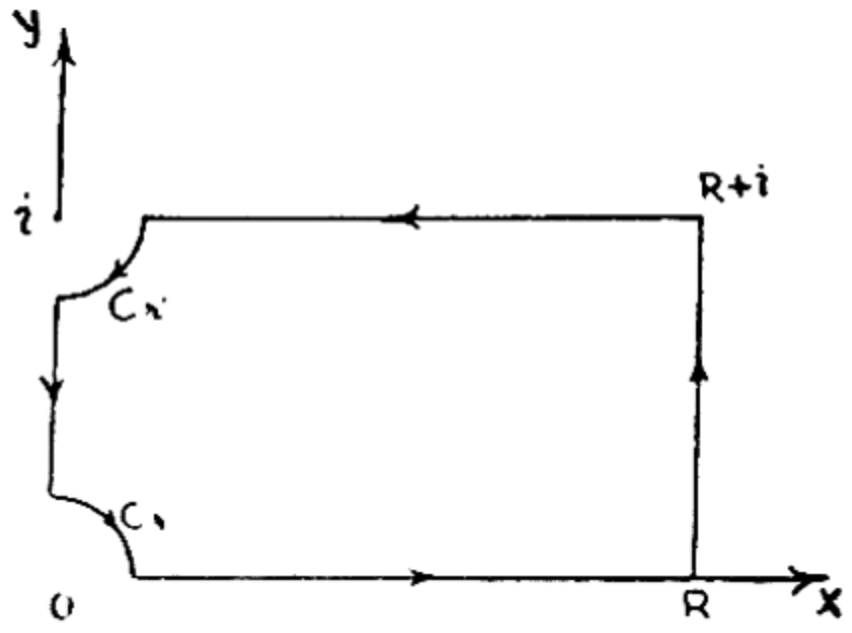


Fig. 35

where r and r' are the radii of indentation at $z=0$ and $z=i$ respectively. $f(z)$ has no singularity within the given contour ; hence, by Cauchy's residue theorem, $I=0$.

Now, $z=0$ and $z=i$ are simple poles of $f(z)$;

$$\therefore \quad \text{residue of } f(z) \text{ at } 0 = \left[\frac{e^{iaz}}{\frac{d}{dz}(e^{2\pi z}-1)} \right]_{z=0} = -\frac{1}{2\pi},$$

and residue at $i = \dfrac{e^{-a}}{2\pi}$ (similarly).

Hence, $\underset{r\to 0}{\text{Lt}} \int_{C_r} f(z)\,dz = \dfrac{2\pi i}{4} \cdot \dfrac{1}{2\pi} = \dfrac{i}{4}$,

and $\underset{r'\to 0}{\text{Lt}} \int_{Cr'} f(z)\,dz = \dfrac{2\pi i}{4} \cdot \dfrac{e^{-a}}{2\pi} = \dfrac{ie^{-a}}{4}$.

Again, $\left| \int_0^1 f(R+iy).\, idy \right| \leqslant \int_0^1 \dfrac{|\, e^{ia(R+iy)} \,|}{|\, e^{2\pi(R+iy)} - 1 \,|} dy$

$$\leqslant \frac{1}{e^{2\pi R} - 1} \int_0^1 e^{-ay}\, dy,$$

which $\to 0$ as $R \to \infty$.

Hence, making $R\to\infty$, $r\to 0$, $r'\to 0$, we have

$$\int_0^\infty \frac{e^{iax}}{e^{2\pi r} - 1} dx - \int_0^\infty \frac{e^{ia(x+i)}}{e^{2\pi(x+i)} - 1} dx - \int_0^1 \frac{e^{ia(iy)}}{e^{2\pi yi} - 1}\, idy = i\frac{e^{-a}+1}{4},$$

$$\text{or } (1-e^{-a}) \int_0^\infty \frac{e^{iax}}{e^{2\pi x} - 1}\, dx = i\frac{1+e^{-a}}{4} + i\int_0^1 \frac{e^{-ay}}{e^{2\pi yi} - 1}\, dy.$$

$$= i\,\frac{1+e^{-a}}{4} + i\int_0^1 \frac{e^{-ay} \cdot e^{-\pi yi}}{e^{\pi yi} - e^{-\pi yi}}\, dy$$

$$= i\frac{1+e^{-a}}{4} + \int_0^1 \frac{e^{-ay}\,(\cos \pi y - i \sin \pi y)}{2 \sin \pi y} dy.$$

Equating imaginary parts we have

$$(1-e^{-a}) \int_0^\infty \frac{\sin ax}{e^{2\pi x} - 1} dx = \frac{1+e^{-a}}{4} + \int_0^1 -\tfrac{1}{2}\, e^{-ay}\, dy$$

$$= \frac{1+e^{-a}}{4} - \frac{1}{2a}\,(1-e^{-a}),$$

$$\text{or } \int_0^\infty \frac{\sin ax}{e^{2\pi x} - 1} dx = \tfrac{1}{4}\,\frac{1+e^{-a}}{1-e^{-a}} - \frac{1}{2a} = \tfrac{1}{4} \coth \frac{a}{2} - \frac{1}{2a}.$$

5.8. Complex Transformation of Known Real Definite Integrals.

By means of a complex change of variable, values of certain integrals can be deduced from known integrals.

Ex. 1. *By integrating* e^{-z^2} *round the rectangle whose vertices are* 0, R, $R+ia$ ia, *where* a *is real and positive, prove that*

$$\text{(i)} \quad \int_0^\infty e^{-x^2} \cos 2ax\, dx = \tfrac{1}{2}\sqrt{\pi}\, e^{-a^2}$$

$$\text{and} \quad \text{(ii)} \quad \int_0^\infty e^{-x^2} \sin 2ax\, dx = e^{-a^2} \int_0^a e^{-y^2}\, dy.$$

(Agra 1946)

We know that $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.

Consider $I = \int_C e^{-z^2}\, dz$, where C is the given contour. The integrand is analytic within and on C; hence, by Cauchy's residue theorem, we have

$$I \equiv \int_0^R e^{-x^2}\, dx + \int_0^a e^{-(R+iy)^2}.\, idy + \int_R^0 e^{-(x+ia)^2} dx + \int_a^0 e^{-(iy)^2} idy = 0.$$

Now, $\left| \int_0^a e^{-(R+iy)^2}.\, idy \right| \leqslant \int_0^a e^{-R^2+y^2}\, dy$, since

$$| e^{-(R+iy)^2} | = e^{-R^2+y^2}$$

i.e., $\leqslant e^{-R^2+a^2}.\, a$, which $\to 0$ as $R \to \infty$.

Hence, $\int_0^\infty e^{-(x+ia)^2} dx = \int_0^\infty e^{-x^2}\, dx - i\int_0^a e^{y^2}\, dy$,

or $\int_0^\infty e^{-x^2+a^2} (\cos 2ax - i \sin 2ax)\, dx$

$$= \frac{\sqrt{\pi}}{2} - i\int_0^a e^{y^2} dy.$$

Equating real and imaginary parts, we have

$$\int_0^\infty e^{-x^2} \cos 2ax\, dx = \frac{\sqrt{\pi}}{2} e^{-a^2},$$

and $$\int_0^\infty e^{-x^2} \sin 2ax\, dx = e^{-a^2} \int_0^a e^{y^2}\, dy.$$

Ex. 2. *Integrating* e^{-z^2} *over the following contour :*

(i) *the* x*—axis from* 0 *to* R,

(ii) *the circle* $|z| = R$, *from* $0 = 0$ *to* $0 = \alpha$, *where* $\alpha \leqslant \frac{\pi}{4}$,

and (iii) *the line* $0 = \alpha$, *from* $|z| = R$ *to the origin* ;
show that

(a) $$\int_0^\infty e^{-x^2\cos 2\alpha} \cos(x^2 \sin 2\alpha) dx = \frac{\sqrt{\pi}}{2} \cos\alpha,$$

and (b) $$\int_0^\infty e^{-x^2\cos 2\alpha} \sin(x^2 \sin 2\alpha)\, dx = \frac{\sqrt{\pi}}{2} \sin\alpha.$$

Deduce (c) $$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Consider $I = \int_C e^{-z^2}\, dz$ round the given contour (see fig. 36). Since the integrand is analytic within and on C, $I \equiv \int_0^R e^{-x^2}\, dx$

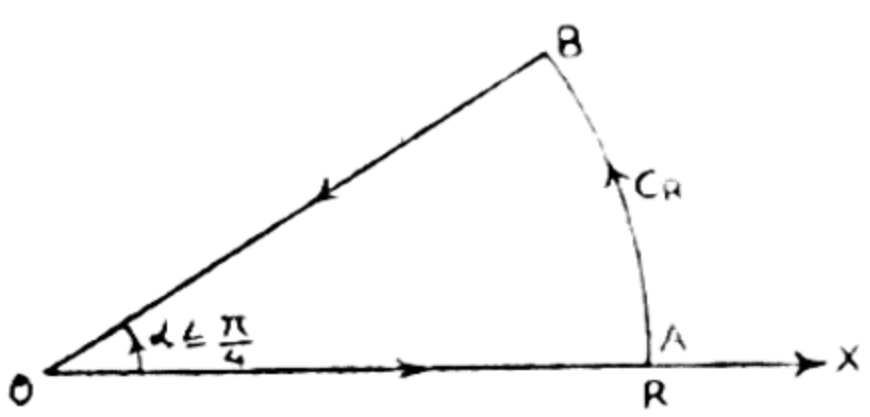


Fig. 36

$$+ \int_{C_R} e^{-z^2} dz + \int_R^0 e^{-(re^{i\alpha})^2} \cdot e^{i\alpha}\, dr = 0.$$

Now, $$\left| \int_{C_R} e^{-z^2} dz \right| \leqslant \int_0^\alpha \left| e^{-R^2 e^{2i\theta}} \right| \left| iRe^{i\theta} \right|$$

$$= R \int_0^\alpha e^{-R^2 \cos 2\theta}\, d\theta = \tfrac{1}{2} R \int_\beta^{\pi/2} e^{-R^2 \sin\phi}\, d\phi,$$

where $\phi = \frac{\pi}{2} - 2\theta$ and $\beta = \frac{\pi}{2} - \alpha \geqslant 0$, if $\alpha \leqslant \frac{\pi}{4}$.

i.e., $\leqslant \frac{R}{2} \int_{\beta}^{\pi/2} e^{-2R^2\rho/\pi} \, d\phi$, by Jordan's inequality.

$= \frac{\pi}{4R} (e^{-2R^2\beta/\pi} - e^{-R^2})$, which $\to 0$ as $R \to \infty$.

Hence, making $R \to \infty$,

$$\int_0^{\infty} e^{-r^2 e^{2ia}} \cdot e^{ia} \, dr = \int_0^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2},$$

or $$\int_0^{\infty} e^{-x^2 \cos 2a} \cdot e^{-ix^2 \sin 2a} \, dx = \frac{\sqrt{\pi}}{2} e^{-ia}.$$

Equating real and imaginary parts, we have

$$\int_0^{\infty} e^{-x^2 \cos 2a} \cos (x^2 \sin 2\alpha) \, dx = \frac{\sqrt{\pi}}{2} \cos \alpha,$$

and $$\int_0^{\infty} e^{-x^2 \cos 2a} \sin (x^2 \sin 2\alpha) \, dx = \frac{\sqrt{\pi}}{2} \sin \alpha.$$

Putting $\alpha = \frac{\pi}{4}$, we have

$$\int_0^{\infty} \cos (x^2) \, dx = \int_0^{\infty} \sin (x^2) \, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}. \qquad \ldots\ldots\ldots (1)$$

Note. The two integrals in (1) are called *Fresnel's integrals.*

5.81. The Converse Problem. So far we have been using Cauchy's residue theorem to evaluate definite integrals. Sometimes we have to use the converse process and find the residue when the value of the integral is known. The example below illustrates the converse process.

Ex. 1. *If n is an even positive integer, find the residue of $\tan^{n-1} \pi z$ at $z = \frac{1}{2}$.* *(Agra 1956)*

Let $f(z)=\tan \pi z=\dfrac{\sin \pi z}{\cos \pi z}$,

then a zero of the denominator is given by $\pi z=2m\pi\pm\dfrac{\pi}{2}$,

or $z=2m\pm\frac{1}{2}(m=0,\pm1,\pm2,\ldots)$;
$\therefore$ $f(z)$ has a simple pole at $z=\frac{1}{2}$. Surround this point by a rectangular contour whose sides are $x=0$, $x=1$, $y=\pm R$ (see fig. 37).

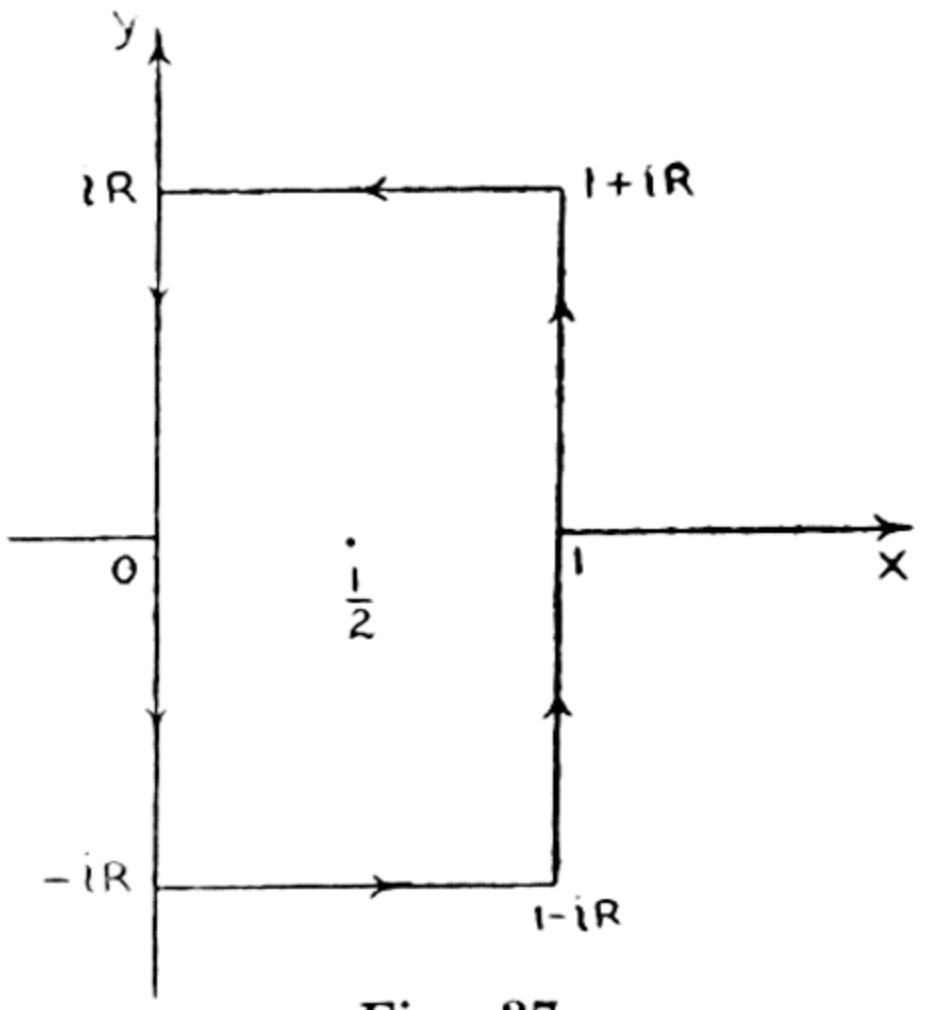


Fig. 37

If k is the residue at $z=\frac{1}{2}$, then, by Cauchy's residue theorem

$$k=\frac{1}{2\pi i}\left[\int_0^1 f(x-iR)\,dx+\int_{-R}^{R} f(1+iy).\,i\,dy+\int_1^0 f(x+iR)\,dx+\int_R^{-R} f(iy).\,i\,dy\right].$$

Now, $f(1+iy)=\tan\pi\,(1+iy)=\tan\pi\,(iy)=f(iy)$;

$$\therefore\quad \int_{-R}^{R} f(1+iy).\,i\,dy+\int_R^{-R} f(iy).\,i\,dy=0.$$

Hence, $$k=\frac{1}{2\pi i}\left[\int_0^1 f(x-iR)\,dx+\int_1^0 f(x+iR)\,dx\right].$$

Since $$\operatorname*{Lt}_{R\to\infty}\int_0^1 f(x-iR)\,dx=\int_0^1 \operatorname*{Lt}_{R\to\infty}\{\tan^{n-1}\pi(x-iR)\}\,dx$$

$$=\int_0^1 \operatorname*{Lt}_{R\to\infty}\left[\frac{1}{i}\cdot\frac{e^{2\pi i(x-iR)}-1}{e^{2\pi i(x-iR)}+1}\right]^{n-1}dx$$

$$=\int_0^1\left(\frac{1}{i}\right)^{n-1}dx=\left(\frac{1}{i}\right)^{n-1},$$

and $$\operatorname*{Lt}_{R\to\infty}\int_1^0 f(x+iR)\,dx=-\int_0^1 \operatorname*{Lt}_{R\to\infty}\left[\frac{1}{i}\cdot\frac{e^{2\pi i(x+iR)}-1}{e^{2\pi i(x+iR)}+1}\right]^{n-1}dx$$

$$=-\int_0^1\left(-\frac{1}{i}\right)^{n-1}dx=(-1)^n\int_0^1\left(\frac{1}{i}\right)^{n-1}dx$$

$$=\left(\frac{1}{i}\right)^{n-1} [\because\ (-1)^n=1].$$

$$\therefore\ k=\frac{1}{2\pi i}\left[\left(\frac{1}{i}\right)^{n-1}+\left(\frac{1}{i}\right)^{n-1}\right]=\frac{1}{\pi i}(-i)^{n-1}=-\frac{i}{\pi}(-i)^{n-1}$$

$$=(-1)^n\,\frac{i^n}{\pi}=\frac{(-1)^{n/2}}{\pi}. \quad [\because\ (-1)^n=1.]$$

5.82. Miscellaneous Solved Examples.

Ex. 1. *Use the residue method to show that*

$$\int_0^{2\pi}\frac{\cos^2 3\theta\, d\theta}{1-2p\cos 2\theta+p^2}=\pi.\frac{1-p+p^2}{1-p},\ 0<p<1.$$

Let $$I=\int_0^{2\pi}\frac{\cos^2 3\theta\, d\theta}{1-2p\cos 2\theta+p^2}=\int_0^{2\pi}\frac{\cos^2 3\theta\, d\theta}{(p-e^{2i\theta})(p-e^{-2i\theta})}.$$

Putting $e^{i\theta}=z$,

$$I=\frac{1}{i}\int_C\frac{z\left\{\frac{1}{2}\left(z^3+\frac{1}{z^3}\right)\right\}}{(p-z^2)(pz^2-1)}dz,$$ where C is the circle $|z|=1$. Within C, the only singularities of the integrand are simples poles at $z=\sqrt{p}$, $z=-\sqrt{p}$ ($\because\ 0<p<1$) and a pole of order 5 at $z=0$.

$$\therefore\ \text{Residue at } \sqrt{p}=-\frac{1}{8i}\,\frac{(p^3+1)^3}{p^3(p^2-1)},$$

$$\left[=\operatorname*{Lt}_{z\to\sqrt{p}}\ \frac{(z-\sqrt{p}).(z^6+1)^2}{4z^5(p-z^2)(pz^2-1)}.\right]$$

$$\text{residue at } -\sqrt{p}=-\frac{1}{8i}\,\frac{(p^3+1)^2}{p^3(p^2-1)},$$

and residue at $z=0$ is the coefficient of $\frac{1}{z}$ in the expansion of the integrand in ascending powers of z which is

$$-\frac{z}{4ip}\left(z^6+2+\frac{1}{z^6}\right)\left(1+\frac{z^2}{p}+\frac{z^4}{p^2}+\dots\right)(1+pz^2+p^2z^4-\dots);$$

hence, residue at $0 = -\frac{1}{4ip}\left(p^2+1+\frac{1}{p^2}\right) = -\frac{1}{4i}\cdot\frac{p^4+p^2+1}{p^2}$.

Hence, $$I = 2\pi i \times \frac{-1}{4ip^3(p^2-1)}\left[(p^3+1)^2+(p^2-1)(p^4+p^2+1)\right]$$

$$= \frac{\pi p^3(p^3+1)}{p^3(1-p^2)} = \frac{\pi(1-p+p^2)}{1-p}.$$

Ex. 2. *Show, by contour integration, that*

$$\int_0^\infty \frac{dx}{(a+bx^2)^n} = \frac{\pi}{2^n b^{\frac{1}{2}}}\cdot\frac{1.3....(2n-3)}{1.2...(n-1)}\cdot\frac{1}{a^{n-\frac{1}{2}}},\ (a>0,\ b>0,\ \text{and}$$

n a positive integer).

Consider $I = \int_C \frac{dz}{(a+bz^2)^n}$, taken round the contour of fig. 28. When R is large, the only singularity of the integrand within the contour is a pole of order n at $z = i\sqrt{\frac{a}{b}}$ $(\because\ a>0,\ b>0)$; and residue thereat is

$$\frac{1}{(\sqrt{b})^n \lfloor n-1}\left[\frac{d^{n-1}}{dz^{n-1}}\left\{\frac{1}{(i\sqrt{a}+\sqrt{b}z)^n}\right\}\right]_{z=i\sqrt{\frac{a}{b}}}$$

$$= \frac{(-1)^{n-1}}{\lfloor n-1}\cdot\frac{n(n+1)\ldots(2n-2)}{b^n\left(2i\sqrt{\frac{a}{b}}\right)^{2n-1}};$$

$$\therefore\ I = \int_{-R}^{R}\frac{dx}{(a+bx^2)^n} + \int_{C_R}\frac{dz}{(a+bz^2)^n}$$

$$= 2\pi i \times \frac{(-1)^{n-1}}{\lfloor n-1}\cdot\frac{\lfloor 2n-2}{\lfloor n-1}\cdot\frac{1}{(2i\sqrt{a})^{2n-1}\,b^{\frac{1}{2}}}$$

$$= \frac{2\pi}{\lfloor n-1}\cdot\frac{1.3.5\ldots(2n-3)\ 2^{n-1}}{2^{2n-1}\,a^{n-\frac{1}{2}}\,b^{\frac{1}{2}}}$$

$$= \frac{\pi}{2^{n-1}\lfloor n-1}\cdot\frac{1.3.5....(2n-3)}{b^{\frac{1}{2}}a^{n-\frac{1}{2}}}.$$

Now, $\left| \int_{C_R} \frac{dz}{(a+bz^2)^n} \right| \leqslant \frac{\pi R}{(bR^2-a)^n} \to 0$ as $R \to \infty$;

$$\therefore \quad \int_{-\infty}^{\infty} \frac{dx}{(a+bx^2)^n} = \frac{\pi}{2^{n-1}\lfloor n-1} \cdot \frac{1.3....(2n-3)}{b^{\frac{1}{2}}\, a^{n-\frac{1}{2}}},$$

$$\text{or} \quad \int_0^{\infty} \frac{dx}{(a+bx^2)^n} = \frac{\pi}{2^n} \cdot \frac{1.3....(2n-3)}{b^{\frac{1}{2}} a^{n-\frac{1}{2}} \lfloor n-1} \cdot \qquad \text{.....(1)}$$

Note. The result will remain true if $a<0$, $b<0$.

Particular Cases.

(i) When $a=b=1$ in (1), we have

$$\int_0^{\infty} \frac{dx}{(1+x^2)^n} = \frac{\pi}{2^n} \cdot \frac{1.3.5....(2n-3)}{\lfloor n-1} \cdot \qquad \text{.....(2)}$$

(ii) Putting $n=1, 2, 3$, etc. in (2), we get

$$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}, \qquad \text{.....(3)}$$

$$\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}, \qquad \text{.....(4)}$$

$$\int_0^{\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{16}, \qquad \text{.....(5) etc.}$$

Ex. 3. *Show that, if* $m>0$, $a>0$,

$$\int_0^{\infty} \frac{x \sin mx}{x^4+a^4}\, dx = \frac{\pi}{2a^2}\, e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}.$$

Let $f(z) = \frac{ze^{imz}}{z^4+a^4}$, and consider $I = \int_C f(z)dz$ taken round the contour of fig. 28. When R is large, the only singularities of $f(z)$ within the contour are two simple poles at $z=ae^{i\pi/4}$ and $z=ae^{3i\pi/4}$, if $a>0$.

$$\text{Residue of } f(z) \text{ at } ae^{i\pi/4} = \left\{ \frac{ze^{imz}}{\frac{d}{dz}(z^4+a^4)} \right\}_{z=ae^{i\pi/4}}$$

$$= \frac{e^{-ma/\sqrt{2}} \cdot e^{ima/\sqrt{2}}}{4ia^2}.$$

Similarly, residue at $ae^{3i\pi/4} = \dfrac{e^{-ma/\sqrt{2}} \cdot e^{-ima/\sqrt{2}}}{-4ia^2}$

Hence, $I \equiv \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$

$$= 2\pi i \times \frac{e^{-ma/\sqrt{2}}}{4ia^2}\left(e^{-ima/\sqrt{2}} - e^{-ima/\sqrt{2}}\right)$$

$$= \frac{i\pi}{a^2} e^{-ma\sqrt{2}} \sin \frac{ma}{\sqrt{2}}.$$

Now, $\left| \int_{C_R} f(z)dz \right| \leqslant \int_{C_R} \frac{|z| \; |e^{imz}|}{|z^4 + a^4|} |dz|$

$$\leqslant \frac{\pi R^2}{R^4 - a^4}, \left[\because |e^{imz}| = e^{-my} \leqslant 1, \text{ for } m > 0 \text{ and } y \geqslant 0. \right]$$

which $\to 0$ as $R \to \infty$;

$$\therefore \int_{-\infty}^{\infty} \frac{xe^{imx}}{x^4 + a^4} dx = \frac{i\pi}{a^2} e^{-ma\sqrt{/2}} \sin \frac{ma}{\sqrt{2}}.$$

Equating imaginary parts, we have

$$\int_0^{\infty} \frac{x \sin mx}{x^4 + a^4} dx = \frac{\pi}{2a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}.$$

Ex. 4. *Prove, by contour integration, that*

$$\int_0^{\infty} \frac{x^3 \sin mx}{x^4 + a^4} dx = \frac{\pi}{2} e^{-ma/\sqrt{2}} \cos \frac{ma}{\sqrt{2}}, \; (a > 0, m > 0).$$

Let $f(z) = \dfrac{z^3 e^{imz}}{z^4 + a^4}$, and consider $I = \int_C f(z)$ round the contour of fig 28. When R is large, the only singularities of $f(z)$ within the contour are two simples poles at $z = ae^{i\pi/4}$ and $z = ae^{3i\pi/4}$, if $a > 0$.

$$\text{Residue of } f(z) \text{ at } ae^{i\pi/4} = \left\{ \frac{z^3 e^{imz}}{\frac{d}{dz}(z^4 + a^4)} \right\}_{z = ae^{i\pi/4}}$$

$$= \tfrac{1}{4} e^{-ma/\sqrt{2}} \cdot e^{ima/\sqrt{2}}.$$

Similarly, residue at $ae^{3i\pi/4} = \frac{1}{4} e^{-ma/\sqrt{2}} e^{-ima/\sqrt{2}}$

Hence, $I \equiv \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$

$$= 2\pi i \times \frac{e^{-ma/\sqrt{2}}}{4} \left(e^{ima/\sqrt{2}} + e^{-iam/\sqrt{2}} \right)$$

$$= \pi e^{-ma/\sqrt{2}} \cos \frac{ma}{\sqrt{2}}.$$

Now, $\left| \int_{C_R} f(z)dz \right| \leqslant \int_{C_R} \frac{|z|^3 |e^{imz}|}{|z^4 + a^4|} |dz|$

$$\leqslant \frac{R^4}{R^4 - a^4} \int_0^{\pi} e^{-mR \sin\theta} d\theta,$$

where $z = Re^{i\theta}$

$$i.\ e., \quad \leqslant \frac{2R^4}{R^4 - a^4} \int_0^{2/\pi} e^{-mR \sin\theta} d\theta,$$

$$\leqslant \frac{2R^4}{R^4 - a^4} \int_0^{\pi/2} e^{-2mR\theta/\pi} d\theta,$$

by Jordan's inequality

$$i.\ e., \quad \leqslant \frac{\pi R^3}{R^4 - a^4} \left| -e^{-2mR\theta/\pi} \right|_0^{\pi/2}$$

$$= \frac{\pi R^3}{R^4 - a^4} (1 - e^{-mR}),$$

which $\to 0$ as $R \to \infty$, if $m > 0$;

$$\therefore \quad \int_{-\infty}^{\infty} \frac{x^3 e^{mix}}{x^4 + a^4} dx = \pi i\, e^{-ma/\sqrt{2}} \cos \frac{ma}{\sqrt{2}}.$$

Equating imaginary parts, we have

$$\int_0^{\infty} \frac{x^3 \sin mx}{x^4 + a^4} dx = \frac{\pi}{2} e^{-ma/\sqrt{2}} \cos \frac{ma}{\sqrt{2}}.$$

Ex. 5. *Prove, by contour integration, that*

$$\int_0^{\infty} \frac{\sin mx}{x(x^2 + a^2)^2} dx = \frac{\pi}{2a^4} - \frac{\pi e^{-ma}}{4a^3} \left(m + \frac{2}{a} \right), \quad (m > 0,\ a > 0).$$

(Agra 1931)

Consider $I=\int_C f(z)dz$, where $f(z)=\dfrac{e^{imz}}{z(z^2+a^2)^2}$, round the contour of fig. 31.

The only singularity of $f(z)$ within the contour is a double pole at $z=ia$.

Putting $z=ia+t$, $f(z)=\dfrac{e^{-am}\cdot e^{imt}}{(ia+t)\ t^2(2ia+t)^2}$

$$=\frac{e^{-am}}{-4a^3i}\left(1+iml-\frac{m^2l^2}{\lfloor 2}-\dots\right)\ \left(1+\frac{t}{ia}\right)^{-1}\left(1+\frac{t}{2ia}\right)^{-2};$$

$\therefore$ residue of $f(z)$ at ia = coefficient of $\dfrac{1}{t}$ in the above expansion $=-\dfrac{e^{-ma}}{4a^4}(am+2)$.

Hence, $I\equiv\int_{-R}^{-r} f(x)dx-\int_{C_r} f(z)dz+\int_r^R f(x)dx$

$$+\int_{C_R} f(z)dz$$

$$=2\pi i\times\frac{-e^{-ma}}{4}(am+2).$$

Now $z=0$ is a simple pole of $f(z)$.

$\therefore$ residue at $0=\underset{z\to 1}{\text{Lt}}\ zf(z)=\underset{z\to 0}{\text{Lt}}\ \dfrac{e^{imz}}{(z^2+a^2)^2}=\dfrac{1}{a^4}$.

Hence, $\underset{r\to 0}{\text{Lt}}\int_{C_r} f(z)dz=\pi i\times\dfrac{1}{a^4}$.

Also, $\left|\int_{C_R} f(z)dz\right|\leqslant\int_{C_R}\dfrac{|e^{imz}|\ |dz|}{|z|\ |z^4+a^4|}$

$$\leqslant\frac{\pi R}{R(R^4-a^4)},$$

$\left[\because |e^{imz}|=e^{-my}\leqslant 1,\ \text{for } m>0, y\geqslant 0.\right]$

which $\to 0$ as $R \to \infty$.

Hence, making $R \to \infty$, $r \to 0$, we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{imx}}{x(x^2+a^2)^2}\,dx - \frac{\pi i}{a^4} = 2\pi i \times -\frac{e^{-ma}}{4a^4}(am+2).$$

Equating imaginary parts, we have

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+x^2)^2}\,dx = \frac{\pi}{a^4} - \frac{\pi e^{-ma}}{2a^4}(am+2),$$

or $$\int_{0}^{\infty} \frac{\sin mx}{x(x^2+a^2)^2}\,dx = \frac{\pi}{2a^4} - \frac{\pi e^{-ma}}{4a^4}(am+2).$$

Note. P. V. is not written in the required result, as $f(x) \to \frac{m}{a^4}$ when $a \to 0$.

Ex. 6. Use the residue method to show that

(i) $$\int_{0}^{\infty} \frac{\log x}{1+x^2}\,dx = 0.$$

(ii) $$\int_{0}^{\infty} \frac{\log (1+x^2)}{1+x^2}\,dx = \pi \log 2.$$

Deduce $$\int_{0}^{1} \frac{\log\left(x+\frac{1}{x}\right)}{1+x^2}\,dx = \frac{\pi}{2}\log 2.$$ *(Agra 1932, '52)*

(i) **Method I.** Consider $I = \int_C f(z)\,dz$ round the contour of fig. 34, where $f(z) = \frac{(\log z)^2}{1+z^7}$. The only singularities of $f(z)$ within the contour are two simple poles at $x = \pm i$;

$\therefore$ residue of $f(z)$ at i, i. e., $e^{i\pi/2} = \left[\frac{(\log z)^2}{\frac{d}{dz}(1+z^2)}\right]_{z=i} = \frac{-\pi^2}{8i}$,

and residue at $-i$, *i.e.*, $e^{3i\pi/2} = \frac{9\pi^2}{8i}$ (similarly).

Hence, $I \equiv \int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_R^r f(xe^{2\pi i}).e^{2\pi i}dx$

$$- \int_{C_r} f(z)dz$$

$$= 2\pi i \times \left(\frac{-\pi^2}{8i} + \frac{9\pi^2}{8}\right) = 2\pi^3.$$

Now, $\left| \int_{C_R} f(z)dz \right| \leqslant \int_{C_R} \frac{|\log z|^2}{|z^2+1|} |dz|$

$$\leqslant \frac{2\pi R(\log R + 2\pi)^2}{R^2-1},$$

$$\left[\because \quad |\log z| = |\log R + i0|, \text{ where } z = Re^{i\theta} \right.$$
$$\left. \leqslant \log R + 2\pi \ (\because \ \theta \leqslant 2\pi). \right]$$

which $\to 0$, as $R \to \infty$.

So also $\left| \int_{C_r} f(z)dz \right| \leqslant \frac{2\pi r \ (\log r + 2\pi)^2}{1-r^2} \to 0$, as $r \to 0$.

Hence, making $R \to \infty$, $r \to 0$, we have

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} dx - \int_0^\infty \frac{\{\log (x\, e^{2\pi i})\}^2}{1+x^2} dx = 2\pi^3,$$

i.e., $$\int_0^\infty \frac{(\log x)^2 - (\log x + 2\pi i)^2}{1+x^2} dx = 2\pi^3.$$

Equating imaginary parts, we get

$$\int_0^\infty \frac{\log x}{1+x^2} dx = 0. \qquad \ldots\ldots(1)$$

Method II. Consider $I_1 = \int_{C_1} \phi(z)dz$, where $\phi(z) = \frac{\log z}{1+z^2}$, round the contour of fig. 31. The only singularity of $\phi(z)$ within the contour is a simple pole at $z=i$, and the residue thereat is equal to

$$\left\{ \frac{\log z}{\frac{d}{dz}(1+z^2)} \right\}_{z=i} = \frac{i\pi/2}{2i} = \frac{\pi}{4}.$$

Hence, $I_1 \equiv \int_R^r e^{\pi i}\phi(x e^{\pi i})dx - \int_{C_r}\phi(z)dz + \int_r^R \phi(x)dx$

$$+\int_{C_R}\phi(z)dz$$

$$= 2\pi i \times \frac{\pi}{4} = \frac{i\pi^2}{2}.$$

Now, $\left|\int_{C_R}\phi(z)dz\right| \leqslant \int_{C_R}\frac{|\log z|}{|z^2+1}\,|dz|$

$$\leqslant \frac{\pi R(\log R+\pi)}{R^2-1},$$

$$\left[\because \quad |\log z| = |\log R + i0|, \text{ where } z = Re^{i\theta} \leqslant \log R+\pi (\because \theta \leqslant \pi).\right]$$

which $\to 0$, as $R\to\infty$.

So also $\left|\int_{C_r}\phi(z)\,dz\right| \leqslant \frac{\pi r\,(\log r+\pi)}{1-r^2} \to 0$ as $r\to 0$.

Hence, making $R\to\infty$, $r\to 0$, we have

$$\int_0^\infty \frac{\log (xe^{\pi i})}{1+x^2}dx + \int_0^\infty \frac{\log x}{1+x^2}\,dx = i\,\frac{\pi^2}{2},$$

i.e., $$\int_0^\infty \frac{(\log x+\pi i)+\log x}{1+x^2}dx = i\frac{\pi^2}{2}.$$

Equating the real parts, we have $\int_0^\infty \frac{\log x}{1+x^2}dx = 0.$

(ii) Consider $I = \int_C f(z)dz$, where $f(z) = \frac{\log (z+i)}{z^2+1}$, round the contour of fig. 28. The only singularity of $f(z)$ within the contour is a simple pole at $z=i$, i.e., $e^{i\pi/2}$ and the residue thereat is

$$\left[\frac{\log (z+i)}{\frac{d}{dz}(z^2+1)}\right]_{z=i} = \frac{\log (2i)}{2i} = \frac{\log 2 + i\,\pi/2}{2i}.$$

Hence, $I \equiv \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \times \frac{\log 2 + i\pi/2}{2i}$

$= \pi(\log 2 + i\pi/2).$

Now, $\left| \int_{C_R} \frac{\log (z+i)}{z^2+1} dz \right| \leqslant \int_{C_R} \frac{|\log (z+i)|}{|z^2+1|} |dz|$

$\leqslant \frac{\pi R\{\log (R+1) + 2\pi\}}{R^2-1},$

$\because \quad |\log (z+i)| = |\log |z+i| + i \text{ amp } (z+i)|$
$\leqslant \log (|z|+1) + 2\pi ;$

$\therefore \quad \left| \int_{C_R} f(z)dz \right| \to 0 \text{ as } R \to \infty.$

Hence, $\int_{-\infty}^{\infty} \frac{\log (x+i)}{x^2+1} dx = \pi (\log 2 + i\pi/2).$

Equating real parts, we get $\int_{-\infty}^{\infty} \frac{\log \sqrt{x^2+1}}{x^2+1} dx = \pi \log 2,$

i.e., $\int_0^{\infty} \frac{\log (x^2+1)}{x^2+1} dx = \pi \log 2. \qquad \ldots\ldots\ldots (2)$

From (1) and (2), we have

$$\pi = \log 2 = \int_0^{\infty} \frac{\log (x^2+1) - \log x}{x^2+1} dx = \int_0^{\infty} \frac{\log (x + \frac{1}{x})}{x^2+1} dx$$

$$= \int_0^1 \frac{\log (x + \frac{1}{x})}{x^2+1} dx + I_1,$$

where $I_1 = \int_1^{\infty} \frac{\log (x + \frac{1}{x})}{x^2+1} dx$

$$=\int_1^{\infty} \frac{\log\left(t+\frac{1}{t}\right)}{1+\frac{1}{t^2}} \cdot \frac{-1}{t^2} dt, \text{ where } x=\frac{1}{t}$$

$$=\int_0^1 \frac{\log\left(t+\frac{1}{t}\right)}{t^2+1} dt.$$

Hence, $\pi \log 2 = 2\int_0^1 \frac{\log\left(x+\frac{1}{x}\right)}{x^2+1} dx,$

$$i.e., \int_0^1 \frac{\log\left(x+\frac{1}{x}\right)}{x^2+1} dx = \frac{\pi}{2}\log 2.$$

Ex. 7. *By integrating* $e^{iz}/\sqrt{z}$ *along a suitable path, show that*

$$\int_0^{\infty} \frac{\cos x}{\sqrt{x}} dx = \int_0^{\infty} \frac{\sin x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}.$$

Consider $I=\int_C \frac{e^{iz}}{\sqrt{z}} dz$

along the contour shown in fig. 38. $f(z)$ has no singularity within the contour ;

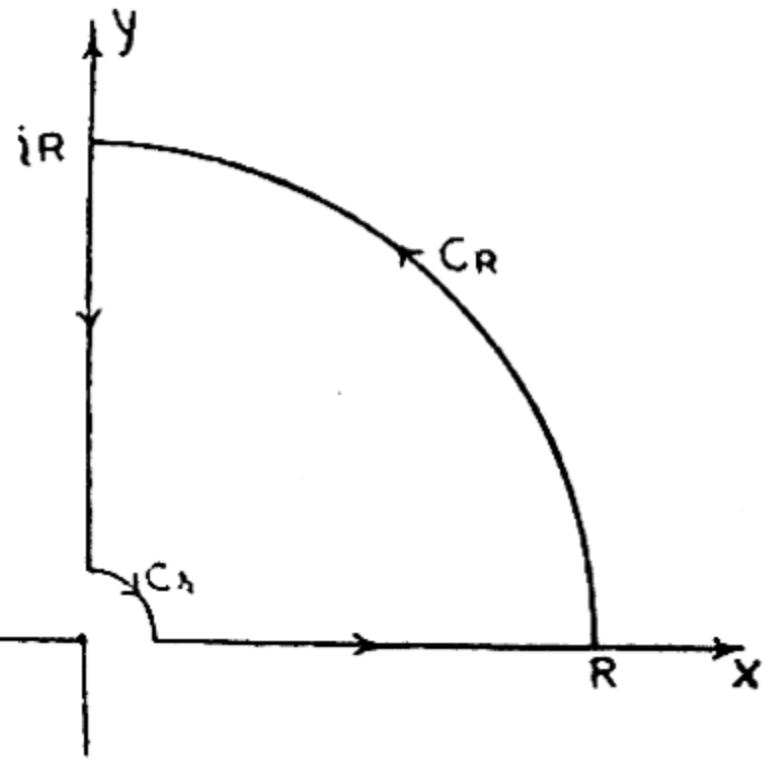


Fig. 38

$$\therefore \quad I \equiv \int_r^R f(x)dx$$

$$+\int_{C_R} f(z)dz+\int_R^0 f(iy).idy-\int_{Cr} f(z)dz=0, \text{ where } f(z)=e^{iz}/\sqrt{z}.$$

$$\text{Now, } \left|\int_{C_R} \frac{e^{iz}}{\sqrt{z}} dz\right| \leqslant \int_{C_R} \frac{|e^{iz}|}{\sqrt{|z|}} |dz|$$

$$=\frac{R}{\sqrt{R}}\int_0^{\pi/2} e^{-R\sin\theta} d\theta, \quad \text{where } z=Re^{i\theta}$$

$$\leqslant \sqrt{R} \int_0^{\pi/2} e^{-\frac{2R}{\pi}\theta} d\theta$$

$$= -\frac{\sqrt{R}}{2R}\pi \Big| e^{-2R\theta/\pi} \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{R}} (1-e^{-R}), \text{ which} \to 0 \text{ as } R \to \infty.$$

Also, $\underset{z\to 0}{\text{Lt}}\ z f(z) = \underset{z\to 0}{\text{Lt}}\ (\sqrt{z}\, e^{iz}) = 0$; $\therefore \int_{Cr} f(z)dz = 0.$

Hence, making $R \to \infty$, $r \to 0$, we have

$$\int_0^\infty \frac{e^{ix}}{\sqrt{x}} dx + \int_\infty^0 \frac{e^{-y}}{\sqrt{iy}} i dy = 0,$$

$$i.e., \int \frac{e^{ix}}{\sqrt{x}} dx = \int_0^\infty \frac{e^{-y}}{\sqrt{iy}} i dy = 2\sqrt{i} \int_0^\infty \frac{e^{-t^2}}{t} t\, dt, \text{ where } \sqrt{y} = t$$

$$= 2e^{i\pi/4} \cdot \frac{\sqrt{\pi}}{2} \left[\because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \cdot \right]$$

$$= \sqrt{\pi}\, e^{i\pi/4}.$$

Equating real and imaginary parts, we have

$$\int_0^\infty \frac{\cos x}{\sqrt{x}} dx = \int_0^\infty \frac{\sin x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}.$$

Ex. 8. *By integrating* $e^{-kz} z^{n-1}$ *round a sector, of radius* R, *bounded by the lines amp* $z=0$, *amp* $z=\alpha < \frac{\pi}{2}$ *or* $> -\frac{\pi}{2}$ *(indented at 0), prove that, if* $k>0$, $n>0$, $\int_0^\infty x^{n-1} e^{-kx\cos\alpha} \begin{matrix}\cos\\ \sin\end{matrix}(kx \sin\alpha)\, dx = k^{-n}\, \Gamma(n) \begin{matrix}\cos\\ \sin\end{matrix} n\alpha$,

where $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$ $(n>0)$. *Show that this formula still holds when* $\alpha = \pm \frac{\pi}{2}$, *if* $n<1$.

Deduce thnt $\int_0^\infty \cos(y^2)\,dy=\int_0^\infty \sin(y^2)\,dy=\left(\frac{1}{8}\pi\right)^{\frac{1}{2}}$.

Let $f(z)=e^{-kn}\ z^{n-1}$ and consider $I=\int f(z)dz$ round the given contour (see fig. 39). $f(z)$ has no sigularity within the contour ;

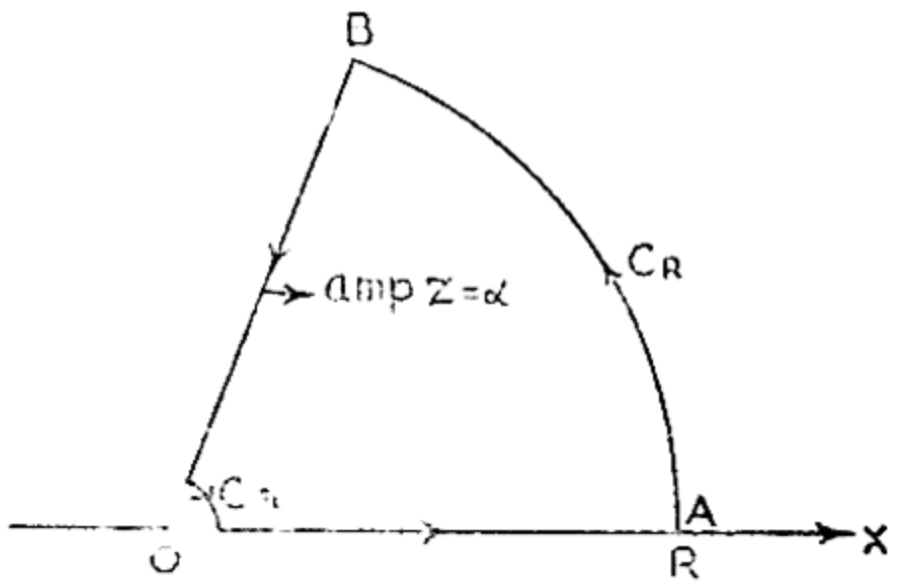


Fig. 39

$$\therefore \quad I\equiv\int_r^R f(x)dx+\int_{C_R} f(z)\,dz+\int_R^0 f(\rho e^{i\alpha}).\,c^{i\alpha}d\rho$$
$$-\int_{Cr} f(z)dz=0.$$

Now, $\left|\int_{Cr} f(z)dz\right| \leqslant r^n\int_0^\alpha e^{-kr\cos\theta}\,d\theta$ where $z=re^{i\theta}$.

$$\leqslant r^n\int_0^\alpha e^{-kr}\,d\theta,\ \text{if}-\pi/2<0<\pi/2$$

i.e., $\leqslant r^n\int_0^\alpha d\theta$, if $k>0$ $(\because r\geqslant 0)$

$=\alpha r^n$, which $\to 0$ as $r\to 0$, if $n>0$;

and $\left|\int_{C_R} f(z)dz\right| \leqslant R^n\int_0^\alpha e^{-kR\cos\theta}\,d\theta$, where $z=Re^{i\theta}$(1)

$$\leqslant R^n\int_{(\frac{\pi}{2}-\alpha)=\lambda}^{\pi/2} e^{-kR\sin\phi}d\phi,$$

where $\theta=\frac{\pi}{2}-\phi$(2)

$$\leqslant R^n\int_\lambda^{\pi/2} e^{-2kR\phi/\pi}\,d\phi$$

$$=\frac{\pi R^{n-1}}{2k}(e^{-2kR\lambda/\pi}-e^{-k\pi}),\qquad \text{.....(3)}$$

which $\to 0$ as $R\to\infty$, if $k>0$ $(\because \lambda>0)$.

Hence, making $R\to\infty$, $r\to 0$, we have

$\int_0^\infty e^{-kx}\, x^{n-1}\, dx + \int_\infty^0 e^{-k\rho e^{i\alpha}} (\rho e^{i\alpha})^{n-1} \cdot e^{i\alpha}\, d\rho = 0$. Now change ρ to x ;

$$\therefore \int_0^\infty e^{-kx\cos\alpha}\, e^{-ikx\sin\alpha} \cdot x^{n-1}\, e^{ni\alpha}\, dx = \int_0^\infty e^{-kx} x^{n-1} dx$$

$$= k^{-n}\int_0^\infty e^{-y}\, y^{n-1}\, dy = k^{-n}\Gamma(n)\ ; \text{ [On putting } kx=y.]$$

$$i.e.,\ \int_0^\infty e^{-kx\cos\alpha}\, e^{-ikx\sin\alpha}\, x^{n-1}\, dx = k^{-n}\,\Gamma(n)\, e^{-ni\alpha}\ .$$

Equating real and imaginary parts, we have

$$\int_0^\infty e^{-kx\cos\alpha} \begin{matrix}\cos\\ \cdot\sin\end{matrix} (kx\sin\alpha).\ x^{n-1}\, dx = k^{-n}\,\Gamma(n).\ \begin{matrix}\cos\\ \sin\end{matrix} n\alpha. \quad \ldots\ldots(4)$$

If $\alpha=\pi/2$, $\lambda=0$ and then $(3)\to 0$ only if $n<1$. Hence, the formula (4) is true when $\alpha=\frac{\pi}{2}$ if $n<1$. Similarly, we can show that the formula (4) holds when $\alpha=-\pi/2$, if $n<1$.

Putting $\alpha=\frac{\pi}{2}$ and $k=1$ in (4), we have

$$\int_0^\infty x^{n-1} \begin{matrix}\cos\\ \sin\end{matrix} x\, dx = \Gamma(n) \begin{matrix}\cos\\ \sin\end{matrix} (n\pi/2). \qquad \ldots\ldots(5)$$

Now, putting $x=y^2$ and $n=\frac{1}{2}$ in (5), we get

$$2\int_0^\infty \begin{matrix}\cos\\ \sin\end{matrix} (y^2)dy = \Gamma(\tfrac{1}{2}) \begin{matrix}\cos\\ \sin\end{matrix} \frac{\pi}{4} = \sqrt{\pi}\cdot\ \frac{1}{\sqrt{2}},$$

$$i.e.,\ \int_0^\infty \begin{matrix}\cos\\ \sin\end{matrix} (y^2)dy = \frac{\sqrt{\pi}}{2\sqrt{2}} = \left(\frac{\pi}{8}\right)^{\frac{1}{2}}.$$

Note. $\underset{r\to 0}{\text{Lt}} \int_{C_r} f(z)dz = 0$ can be shown as follows also :

$$\underset{z\to 0}{\text{Lt}}\ zf(z) = \underset{z\to 0}{\text{Lt}}\ z\,(e^{-kz}\, z^{n-1}) = \underset{L\to 0}{\text{Lt}}\ (z^n e^{-kz})$$

$=0$, if $n>0$. Hence etc.

Exercise 5

1. Show that the residues of $\frac{e^{iz}}{z^2+a^2}$ and $\frac{ze^{iz}}{z^2+a^2}$ at ai are $\frac{e^{-a}}{2ai}$ and $\frac{e^{-a}}{2}$ respectively.

2. If k is any integer, show that the residue of $\cot z$, at $k\pi$ is 1.

3. Prove that the residue of the rational function $\frac{P(z)}{Q(z)}$ at a double pole $z=a$ is

$$\frac{1}{40\{Q'''(a)\}^3}\left[\begin{array}{r} 120\,P''(a)\{Q'''(a)\}^2-60\,P'(a)\,Q'''(a)\,Q^{iv}(a) \\ -12\,P(a)\,Q'''(a)\,Q^{v}(a)+15\,P(a)Q^{iv}(a)\end{array}\right]$$

4. If in the rational function $\frac{P(z)}{Q(z)}$, $P(z)$ has a simple zero at $z=a$ and $Q(z)$ a double zero at $z=a$, show that the residue of the function at a is $2\,P'(a)/Q''(a)$.

5. Show that the sum of the residues of any rational function is zero.

6. If $f(z)=\sum_{r=1}^{n}\frac{A_r}{(z-a)^r}$, show that the residue of $\frac{f(z)}{z-\zeta}$ at a is $-f(\zeta)$.

7. If $f(z)=\sum_{r=1}^{n} A_r/(z-a)^r+\phi(z)$, where $\phi(z)$ is analytic at a, show that the residue of $f(z)/(z-z_1)$ at a is $-\sum_{r=1}^{n} A_r\,(z_1-a)^{-r}$.

Use the method of contour integration to prove the following results (8—47) :

8. $\int_0^{2\pi}\frac{\sin^2\theta d\theta}{a+b\cos\theta}=\frac{2\pi}{b^2}(a-\sqrt{a^2-b^2})$, $a>b>0$.

Will this result be true when $a=b$? Give reasons.

9. $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)=\frac{2\pi}{\lfloor n}$, where n is a positive integer,

10. $\int_0^{2\pi}\frac{d\theta}{1-2p\cos\theta+p^2}=\frac{2\pi}{1-p^2}$, $(0\leqslant p<1)$.

11. $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$, $a>b>0$.

12. $\int_0^{9\pi} \frac{d\theta}{(a+b\cos^2\theta)^2} = \frac{\pi(2a+b)}{a^{3/2}(a+b)^{3/2}}$, $a>b>0$.

13. $\int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$, ($n$ a positive integer).

14. $\int_0^{2\pi} \cos^n d\theta = 0$, if n is odd, and

$$= \frac{1.3.5....(2n-1)}{2.4.6....n} 2\pi, \text{ if } n \text{ is even.}$$

15. $\int_0^{2\pi} \frac{d\theta}{1-ae^{i\theta}} = 2\pi$, or 0, according as $|a|<1$, or $|a|>|$

16. $\int_0^{\infty} \frac{\cos x}{x^2+a^2} dx = \frac{\pi e^{-a}}{2a}$, $(a>0)$, *(Agra 1936,'39,'50)*

17. $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx = \frac{5\pi}{12}$.

18. $\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$, $(a>0)$.

19. (i) $\int_0^{\infty} \frac{x^2 dx}{(x^2+1)^3} = \frac{\pi}{16}$; (ii) $\int_0^{\infty} \frac{x^4 dx}{(1+x^2)^4} = \frac{\pi}{32}$.

20. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)\,(x^2+c^2)^2} = \frac{\pi\,(b+2c)}{2bc^3\,(b+c)^2}$, $(b>0,\ c>0)$.

21. $\int_0^{\infty} \frac{x^6 dx}{(a^4+x^4)^2} = \frac{3\sqrt{2}}{16a}\pi$, $(a>0)$.

22. $\int_0^{\infty} \frac{\cos ax}{1+x^2+x^4} dx = \frac{\pi}{\sqrt{3}} \sin\tfrac{1}{2}\left(a+\frac{\pi}{3}\right) . e^{-a\sqrt{3}/2}$, $(a>0)$.

23. $\int_0^{\infty} \frac{\cos mx}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3}(1+am)e^{-am}$, $(a>0,\ m>0)$.

24. $\int_0^\infty \frac{x^{2m+1}}{x^{2n}+1}dx = \frac{\pi}{2n \sin\left(\frac{2m+1}{2n}\pi\right)}$, where m and n are positive integers and $m<n$.

25. $\int_0^\infty \frac{x^2 dx}{(x^4+a^4)^3}dx = \frac{5\pi\sqrt{2}}{128\, a^9}$, $(a>0)$.

26. $\int_0^\infty \frac{\cos^2 x\, dx}{(1+x^2)^2} = \frac{\pi}{8}(1+3e^{-2})$.

27. (i) $\int_0^\infty \frac{(1+x^2)\cos ax}{1+x^2+x^4}dx = \frac{\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \cos\frac{a}{2}$;

[**Hint :** Integrate $\frac{e^{iaz}}{1+z+z^2}$ round a semicircle]

(ii) $\int_0^\infty \frac{x \sin ax}{1+x^2+^4}dx = \frac{\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \sin\frac{a}{2}$, where $a>0$.

28. $\int_0^\infty \frac{\cos ax}{(1+x^2)^3}dx = \frac{\pi e^{-a}}{16}\cdot (a^2+3a+3)$, $(a>0)$.

29. $\int_0^\infty \frac{\sin^2 mx}{x^2(x^2+a^2)}dx = \frac{\pi}{4a^3}(e^{-2ma}-1+2ma)$, $(m>0,\ a>0)$.

[**Hint :** Integrate $\frac{1-e^{2imz}}{z^2(z^3+a^2)}$ round the contour of fig. 31]

30. $\int_0^\infty \frac{\sin x}{x(x^2+a^2)}dx = \frac{\pi}{2a^2}(1-e^{-a})$, $(a>0)$. *(Agra, 1931)*

31. (i) P.V. $\int_0^\infty \frac{\cos ax}{x^2-r^2}dx = -\frac{\pi}{2r}\sin ar$;

(ii) P.V. $\int_0^\infty \frac{x \sin ax}{x^2-r^2}dx = \frac{\pi}{2}\cdot\cos ar$; $(a>0,\ r$ real).

[**Hint :** Integrate $\frac{e^{iaz}}{z-r}$ round a semi-circle indented at $z=r$.]

32. $\displaystyle\int_0^\infty \frac{\sin \pi a}{x(1-x^2)}dx=\pi.$

33. P.V. $\displaystyle\int_0^\infty \frac{x^4}{x^6-1}dx=\frac{\pi\sqrt{3}}{6}.$

34. $\displaystyle\int_0^\infty \frac{x-\sin x}{x^3(a^2+x^2)}dx=\frac{\pi}{2a^4}\left(\tfrac{1}{2}a^2-a+1-e^{-a}\right),\ (a>0).$

35. $\displaystyle\int_0^\infty \frac{\log x}{(x^2+1)^2}dx=-\frac{\pi}{4}.$

36. $\displaystyle\int_0^\infty \frac{\sin^3 x\cos x}{x^3}dx=\frac{\pi}{4}.$

37. $\displaystyle\int_{-\infty}^\infty \frac{\sin m\,(x-a)}{x-a}\cdot\frac{\sin n\,(x-b)}{x-b}dx=\pi\frac{\sin n\,(a-b)}{a-b}$, where a and b are real, and $m\geqslant n\geqslant 0$.

38. $\displaystyle\int_0^\infty \frac{\sin^2 x\sin ax}{a^3}dx=\tfrac{1}{2}\pi a-\tfrac{1}{8}\pi a^2$, where $0<a<2$.

39. $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,\frac{\sin nx}{\sin x}dx=\frac{\pi(e^n-1)}{(e^2-1)e^{n-1}}$ where n is a positive integer.

40. P.V. $\displaystyle\int_0^\infty \frac{\tan x}{x}dx=\tfrac{1}{2}\pi.$

41. $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^2}dx=\pi/\left(2\sin\frac{\pi a}{2}\right),\ (0<a<2).$

42. $\displaystyle\int_0^\infty \frac{e^{p-1}}{x+e^{i\lambda}}dx=\frac{\pi e^{i(p-1)\lambda}}{\sin \pi p},\ (0<p<1,\ -\pi<\lambda<\pi).$

43. $\displaystyle\int_0^\infty \frac{x^p}{(1+x^2)^2}dx=\frac{\pi}{4}(1-p)\sec\frac{\pi p}{2},\ (-1<p<3).$

44. $\displaystyle\int \frac{(\log x)^2}{1+x+x^2}dx=\frac{16}{81\sqrt{3}}\pi^3.$ **[Hint :** Integrate $\displaystyle\frac{(\log z)^3}{1+z+z^2}$ round the contour of fig. 34.]

45. $\displaystyle\int_0^\infty \frac{x^{a-1}\log x}{1+x}dx=-\pi^2\cot \pi a\operatorname{cosec}\pi a,\ (0<a<1).$

46. (i) $\int_0^\infty \frac{x^{a-1}}{1+x+x^2}dx=\frac{2\pi}{\sqrt{3}}\cos\frac{2\pi a+\pi}{6}\operatorname{cosec}\pi a$;

(ii) $\int_0^\infty \frac{x^{a-1}}{1-x+x^2}dx=\frac{2\pi}{\sqrt{3}}\sin\frac{2\pi a+\pi}{3}\operatorname{cosec}\pi a;\ (0<a<2)$

[**Hint :** Integrate $\frac{z^{a-1}}{1+z+z^2}$ round the contour of fig. 31]

47. $\int_0^\infty \frac{\cos x^2+\sin x^2-1}{x^2}dx=0.$

[**Hint :** Integrate $\frac{e^{iz^2}-1}{z^2}$ round a quadrant of a circle indented at 0.]

48. (i) $\int_0^\infty \frac{\sin ax}{\sinh x}dx=\frac{\pi}{2}\tanh\frac{a\pi}{2}$:

(ii) $\int_0^\infty \frac{x\cos ax}{\sinh x}dx=\frac{\pi^2 e^{-a\pi}}{(1+e^{-a\pi})^2}$; where a is real.

49. Integrate $\frac{\sqrt{z}\log z}{(1+z)^2}$, where $0<\text{amp } z<2\pi$, round the contour of fig. 34. and show that

(i) $\int_0^\infty \frac{\sqrt{x}\log x}{(1+x)^2}dx=\pi,$

(ii) $\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}dx=\frac{\pi}{2}.$

50. (i) Integrate $\frac{e^{iz}}{z-ia}$, where $a>0$, over the semi-circular contour of fig. 28 and show that

$$\int_{-\infty}^{\infty} \frac{a\cos x+x\sin x}{x^2+a^2}dx=2\pi e^{-a}.$$

(ii) By integrating $\frac{e^{iz}}{z+ia}$, where $a>0$, prove that

$$\int_{-\infty}^{\infty} \frac{-a\cos x+x\sin x}{x^2+a^2}dx=0.$$

51. Integrate $\frac{z \log (1-iz)}{(1+2z^2)^2}$ round the semi-circle of fig. 28, and show that

$$\int_0^\infty \frac{x \tan^{-1} x}{(1+2x^2)^2}\,dx = \int_0^1 \frac{x \sin^{-1} x}{(1+x^2)^2}\,dx = \frac{\pi}{8}\,(\sqrt{2}-1).$$

52. By integrating $\frac{(1-r)+(1+r)\,z^2}{(1-r)^2+(1+r)^2\,z^2}\,\frac{\log z}{1+z^2}$ round the contour of fig. 31, show that $\int_0^{\pi/2} \frac{1-r\cos 2\theta}{1-2r\cos 2\theta+r^2} \log\tan\theta\,d\theta$

$$= \frac{\pi}{4}\log\left(\frac{1-r}{1+r}\right), \text{ if } -1<r<1,$$

and

$$= -\frac{\pi}{4}\log\left(\frac{r-1}{r+1}\right), \text{ if } |r|>1.$$

53. If $b>0$, $r>0$, and $0<a<2$, show that

$$\int_0^\infty x^{a-1}\, e^{\cos bx} \sin\left(\frac{a\pi}{2}-\sin bx\right) \frac{dx}{x^2+r^2} = \frac{\pi}{2}\, r^{a-2}\, e^{e^{-br}}.$$

54. By integrating round a suitable contour, show that

$$\int_0^\infty \frac{\sin x^2}{x}\,dx = \frac{\pi}{4}. \text{ Deduce } \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

55. By integrating $e^{iz}\, z^{a-1}$ round a quardrant of a circle, prove that, if $0<a<1$, $\int_0^\infty x^{a-1} \begin{matrix}\cos\\ \sin\end{matrix} x\,dx = \Gamma(a) \begin{matrix}\cos\\ \sin\end{matrix} \frac{\pi a}{2}$.

56. By integrating $\frac{e^{az}}{\sin h\pi z}$, where $-\pi<a<\pi$, round the rectangle of sides $y=0$, $y=1$, $x=\pm R$, indented at 0 and i, show that

$$\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\,dx = \tfrac{1}{2}\tan\frac{a}{2}.$$

57. Integrate $\frac{e^{iz}}{z+a}$, where $a>0$, round the square whose sides are $x=0$, $x=R$, $y=0$, $y=R$, and show that

(i) $\int_0^\infty \frac{\cos x}{x+a}\,dx = \int_0^\infty \frac{xe^{-ax}\,dx}{1+x^2}$;

(ii) $\int_0^\infty \frac{\sin x}{x+a}dx = \int_0^\infty \frac{e^{-ax}}{1+x^2}\,dx$.

58. Integrate $\frac{z}{p-e^{-iz}}$ round the rectangle of sides $x=\pm\pi$, $y=0$, $y=R$, and show that

$$\int_0^\pi \frac{x \sin x}{p^2-2p\cos x+1}dx = \frac{\pi}{p}\log(1+p), \text{ if } 0<p<1,$$

$$= \frac{\pi}{p}\log\left(1+\frac{1}{p}\right), \text{ if } p>1.$$

(Agra 1937,'40,'42,'55)

59. By integrating $\frac{z}{\sinh z}$ round a suitable contour, show that

$$\int_0^\infty \frac{x}{\sinh x}dx = \frac{\pi^2}{4}.$$

60. If $-2\pi<c<2\pi$, show that $\int_0^\infty \frac{\cosh cx}{\cosh^2 \pi x}dx = \frac{c}{2\pi}\operatorname{cosec}\frac{c}{2}$.

61. Prove that $\int_0^\infty \frac{\sin ax}{e^{2\pi x}+1}dx = \frac{1}{2a}-\frac{1}{4\sinh a/2}$, where a is real.

62. Integrate e^{-z^2} round the rectangle of sides $y=0$, $y=b$, $x=\pm R$, and show that $\int_{-\infty}^\infty e^{-(x+bi)^2}dx = \int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$.

Deduce (i) $\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$;

(ii) $\int_0^\infty e^{-(x+c)^2}dx = \sqrt{\pi}$, where c is any constant.

63. By integrating $\frac{e^{ipz}}{\cosh^2 z}$ along the lines $y=0$, $y=\pi$, prove that

$$\int_0^\infty \frac{\cos px}{\cosh^2 x}dx = \frac{\pi p}{2\sinh \frac{1}{2}\pi p}.$$

APPENDIX

Expansion of Meromorphic Functions

If a function is analytic in a region except at a finite number of poles, it is said to be *meromorphic* in the region.

A rational function is the simplest example of a meromorphic function. We already know how to resolve a rational function into partial fractions. We shall now find a similar expression for more general meromorphic functions.

Let $f(z)$ be a function whose only singularities, except at infinity, are simple poles $a_1, a_2, a_3, \ldots$, where $0 < |a_1| \leqslant |a_2| \leqslant |a_3| \leqslant \ldots$ and let the residues at these poles be $b_1, b_2, b_3, \ldots$ respectively. If there is a sequence of closed contours $\{C_n\}$, such that :

(i) C_n *encloses* $a_1, a_2, a_3, \ldots, a_n$ *but no other poles ;*

(ii) the minimum distance R_n *of* C_n *from the origin tends to infinity as* $n \to \infty$;

(iii) the length L_n *of* C_n *is* $O(R_n)$*

and (iv) on C_n, $f(z) = o(R_n)$* *which condition is satisfied if* $f(z)$ *is bounded on the sequence of contours* $\{C_n\}$;

then for all values of z except the poles

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z-a_n} + \frac{1}{a_n} \right).$$

Let $I = \dfrac{1}{2\pi i} \displaystyle\int_{C_n} \frac{f(w)}{w(w-z)}\,dw,$

where z is a point within C_n. The integrand has simple poles at the points a_m, 0, z and the residues at these points are $\dfrac{b_m}{a_m(a_m-z)}$, $-\dfrac{f(0)}{z}$ and $\dfrac{f(z)}{z}$ respectively. In particular cases the last two residues may vanish. Hence,

$$I=\sum_{m=1}^{n}\frac{b_m}{a_m(a_m-z)}-\frac{f(0)}{z}+\frac{f(z)}{z}.$$

Now, $|I|=\left|\frac{1}{2\pi i}\int_{C_n}\frac{f(w)}{w(w-z)}\,dw\right|$

$$\leqslant\frac{L_n}{2\pi\, R_n(R_n-|z|)}M_n,$$

where $|f(w)|\leqslant M_n$ on C_n;

$\therefore\ |I|\to 0$ as $n\to\infty$, because of the given condition.

Hence, $\underset{n\to\infty}{\text{Lt}}\ \sum_{m=1}^{n}\frac{b_m}{a_m(a_m-z)}-\frac{f(0)}{z}+\frac{f(z)}{z}=0,$

i.e., $f(z)=f(0)+\sum_{n=1}^{\infty}b_n\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right),$(1)

which is the required result.

Note 1. It is obvious from the proof that the series (1) converges uniformly in any finite region which does not contain any of the poles.

***Note 2.** *The 'o,O' notation.* Let the functions $f(n)$ and $\phi(n)$ be defined for all sufficiently large values of n and let them tend to zero or to infinity as $n\to\infty$. If $\frac{f(n)}{\phi(n)}\to 0$ as $n\to\infty$, $f(n)$ is said to be an infinitesimal with respect to $\phi(n)$ and we write $f(n)=o[\phi n])$. If $\frac{f(n)}{\phi(n)}\to k\neq 0$, $f(n)$ and $\phi(n)$ are said to be of the same order and we write $f(n)=O[\phi(n)]$.

Thus, the function $2n+3n^2$ is $o(n^3)$ and $O(n^2)$ as $n\to\infty$.

The 'o, O' notation is without any ambiguity only when the limit to which the variable tends is known. For example, the function $2x+3x^2$ is $O(x)$ when $x\to 0$, but $O(x^2)$ when $x\to\infty$.

Ex. 1. *Prove that*

$$\sec z=4\pi\sum_{0}^{\infty}(-1)^n(2n+1)/\{(2n+1)^2\pi^2-4z^2\}.\qquad (Agra\ 1960)$$

Since $\cos z$ has simple zeros at the points $z=\pm(2n+1)\frac{\pi}{2}$, $n=0, 1, 2, 3,\ldots$; $\therefore$ $\sec z$ has simple poles at these points.

The residue at $(n+\frac{1}{2})\pi = \left[\frac{1}{\frac{d}{dz}(\cos z)}\right]_{z=(n+\frac{1}{2})\pi}$

$$=(-1)^{n+1}.$$

Similarly, the residue at $-(n+\frac{1}{2})\pi=(-1)^n$.

Let C_n be the square $ABCD$ with corners at the points $(n+1)(\pm1\pm i)\pi$ [see fig. 40], then:

(i) C_n encloses the poles $z=\pm(m+\frac{1}{2})\pi$, $m=0,1,2,\ldots,n$, and no other poles ;

(ii) the minimum distance R_n of C_n from the origin is $(n+1)\pi$ which tends to infinity as $n\to\infty$;

(iii) the length L_n of C_n is $8(n+1)\pi$ and is, therefore, equal to 8 R_n, *i.e.*, $L_n=O(R_n)$;

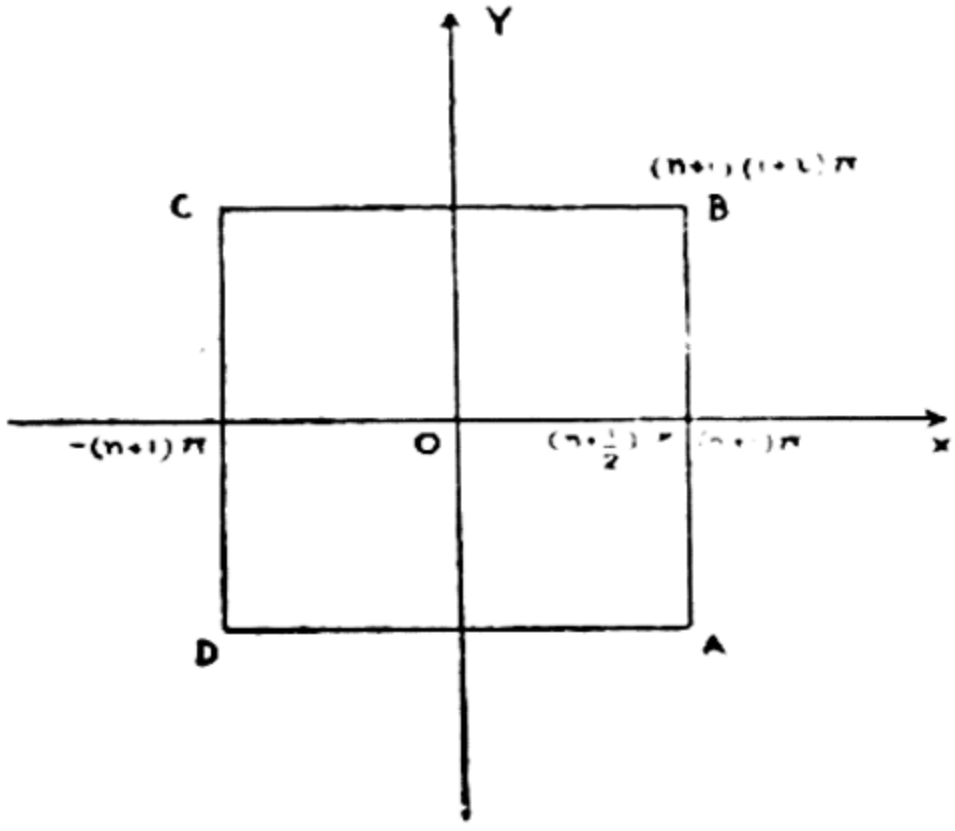


Fig. 40

and (iv) $f(z)$ is bounded on C_n as shown below.

Along AB, $x=(n+1)\pi$;

$$\therefore \quad |\sec z| = \left|\frac{2}{e^{iz}+e^{-iz}}\right| = 2/\left|e^{i(n+1)\pi}.\,e^{-y}+e^{-i(n+1)\pi}.\,e^{y}\right|$$

$$\leqslant 2/(e^{-y}+e^{y})\leqslant 1.$$

Similarly, along CD, $|\sec z|\leqslant 1$.

Again, $|\sec z| = 2/|e^{iz}+e^{-iz}| \leqslant \left|\frac{2}{|e^{iz}|-|e^{-iz}|}\right|$,

i. e., $\leqslant 2/|e^{-y}-e^{y}|$;

$\therefore$ along BC or AD, $|\sec z|\leqslant 2/|e^{-(n+1)\pi}-e^{(n+1)\pi}|$.

Thus, $\sec z$ is bounded on C_n.

Hence, $\sec z = \sec(0)$

$$+\sum_0^\infty (-1)^{n+1}\left[\frac{1}{z-(n+\frac{1}{2})\pi}+\frac{1}{(n+\frac{1}{2})\pi}\right]$$

$$+\sum_0^\infty (-1)^n\left[\frac{1}{z+(n+\frac{1}{2})\pi}-\frac{1}{(n+\frac{1}{2})\pi}\right]$$

$$= 1+2\sum_0^\infty (-1)^n\left[\frac{1}{(2n+1)\pi-2z}+\frac{1}{(2n+1)\pi+2z}\right]$$

$$+2\sum_0^\infty (-1)^{n+1}\frac{2}{(2n+1)\pi}$$

$$=1+4\pi\sum_0^\infty (-1)^n\frac{2n+1}{(2n+1)^2\pi^2-4z^2}-\frac{4}{\pi}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots)$$

$$= 4\pi\sum_0^\infty (-1)^n\frac{2n+1}{(2n+1)^2\pi^2-4z^2}\cdot\left[\because 1-\tfrac{1}{3}+\tfrac{1}{5}-\ldots=\frac{\pi}{4}\right]$$

Ex. 2. Prove that

(i) $\operatorname{cosec} z=\frac{1}{z}+2z\sum_1^\infty \frac{(-1)^{n-1}}{n^2\pi^2-z^2}$.

(ii) $\tan z = 8z\sum_0^\infty \frac{1}{(2n+1)^2\pi^2-4z^2}$.

(iii) $\cot z=\frac{1}{z}+2z\sum_1^\infty \frac{1}{z^2-n^2\pi^2}$.

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